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Summation of a special ${}_4F_3$

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Summary. - *It is shown that*

$$ {}_4F_3\left[\begin{matrix} -n, \frac{1}{2}(\alpha+1), \frac{1}{2}\alpha+1, \lambda+n; \\ \alpha+1, \frac{1}{2}(\lambda+1), \frac{1}{2}\lambda+1 \end{matrix}\right] = \frac{\lambda(\lambda-\alpha)_n}{(\lambda+2n)(\lambda)_n}, $$

1. Consider the sum

$$ A_n(\alpha, \lambda) = \sum_{r=0}^{n} \frac{(-n)_r(\alpha+1)_{2r}(\lambda+n)_r}{r!\,(\alpha+1)_r(\lambda+1)_{2r}} $$

$$ = {}_4F_3\left[\begin{matrix} -n, \frac{1}{2}(\alpha+1), \frac{1}{2}\alpha+1, \lambda+n; \\ \alpha+1, \frac{1}{2}(\lambda+1), \frac{1}{2}\lambda+1 \end{matrix}\right], $$

which incidentally is Saalschützian. We shall show that

$$ A_n(\alpha, \lambda) = \frac{\lambda(\lambda-\alpha)_n}{(\lambda+2n)(\lambda)_n}. \tag{1} $$

If we put

$$ f_n(\alpha, \lambda) = \sum_{r=0}^{n} (-1)^r \binom{n}{r} (\alpha+r+1)_r(\lambda+2r+1)_{n-r-1}(\lambda+2n), $$

where it is understood that the term on the right corresponding to $r=n$ is

$$ (-1)^n(\alpha+n+1)_n, $$

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then it is easily verified that (1) is equivalent to

$$f_n(\alpha, \lambda) = (\lambda - \alpha)_n . \tag{2}$$

Now we have

$$f_n(\alpha + 1, \lambda) - f_n(\alpha, \lambda)$$

$$= \sum_{r=0}^{n} (-1)^r \binom{n}{r} \{ (\alpha + r + 2)_r - (\alpha + r + 1)_r \} \cdot (\lambda + 2r + 1)_{n-r-1}(\lambda + 2n)$$

$$= \sum_{r=0}^{n} (-1)^r \binom{n}{r} r(\alpha + r + 2)_{r-1}(\lambda + 2r + 1)_{n-r-1}(\lambda + 2n)$$

$$= - n \sum_{r=0}^{n-1} (-1)^r \binom{n-1}{r} (\alpha + r + 3)_r(\lambda + 2r + 3)_{n-r-2}(\lambda + 2n)$$

which evidently implies

$$f_n(\alpha + 1, \lambda) - f_n(\alpha, \lambda) = - n f_{n-1}(\alpha + 2, \lambda + 2). \tag{3}$$

Clearly (2) holds when $n = 0$. Assuming the truth of (2) for all values less than n, it follows from (3) that

$$f_n(\alpha + 1, \lambda) - f_n(\alpha, \lambda) = - n(\lambda - \alpha)_{n-1} . \tag{(4}$$

If we keep λ fixed and note that $f_n(\alpha, \lambda)$ is a polynomial in α, it is clear from (4) that

$$f_n(\alpha, \lambda) = (\lambda - \alpha)_n + g_n(\lambda), \tag{5}$$

where $g_n(\lambda)$ is independent of α. We now take $\alpha = \lambda$, so that (5) becomes

$$f_n(\lambda, \lambda) = g_n(\lambda) \qquad (n > 0).$$

But

$$f_n(\lambda, \lambda) = \frac{(\lambda + 2n)(\lambda)_n}{\lambda} A_n(\lambda, \lambda)$$

$$= \frac{(\lambda + 2n)(\lambda)_n}{\lambda} \sum_{r=0}^{n} \frac{(-n)_r(\lambda + n)_r}{r!\,(\lambda + 1)_r}$$

and

$$\sum_{r=0}^{n} \frac{(-n)_r(\lambda + n)_r}{r!\,(\lambda + 1)_r} = \frac{(1 - n)_n}{(\lambda + 1)_n} = 0$$

for $n > 0$. Hence $g_n(\lambda) = 0$ and (5) reduces to (2).

2. We have

$$\sum_{r=0}^{\infty} \frac{(\lambda + 2n)(\lambda)_n}{n!} A_n(\alpha, \lambda)t^n$$

$$= \sum_{n=0}^{\infty} \frac{(\lambda + 2n)(\lambda)_n}{n!} t^n \sum_{r=0}^{\infty} (-1)^r \binom{n}{r} \frac{(\alpha + 1)_{2r}(\lambda + n)_r}{(\alpha + 1)_r(\lambda + 1)_{2r}}$$

$$= \sum_{r=0}^{\infty} \frac{(\alpha + 1)_{2r}t^r}{r!\,(\alpha + 1)_r(\lambda + 1)_{2r}} \sum_{n=r}^{\infty} \frac{t^{n-r}}{(n - r)!} (\lambda + 2n)(\lambda)_{n+r}$$

$$= \sum_{r=0}^{\infty} \frac{(\alpha + 1)_{2r}(\lambda)_{2r}t^r}{r!\,(\alpha + 1)_r(\lambda + 1)_{2r}} \sum_{n=0}^{\infty} \frac{t^n}{n!} (\lambda + 2r + 2n)(\lambda + 2r)_n$$

$$= \sum_{r=0}^{\infty} \frac{(\alpha + 1)_{2r}t^r}{r!\,(\alpha + 1)_r(\lambda + 2r)} \Big\{ (\lambda + 2r) \sum_{n=0}^{\infty} \frac{(\lambda + 2r)_n}{n!} t^n$$

$$+ 2t(\lambda + 2r) \sum_{n=0}^{\infty} \frac{(\lambda + 2r + 1)_n}{n!} t^n \Big\}$$

$$= \lambda \sum_{r=0}^{\infty} \frac{(\alpha+1)_{2r} t^r}{r!\,(\alpha+1)_r} \{ (1-t)^{-\lambda-2r} + 2t(1-t)^{-\lambda-2r-1} \} .$$

It accordingly follows from (1) or (2) that

$$\lambda \sum_{n=0}^{\infty} \frac{(\alpha+1)_{2r} t^r}{r!\,(\alpha+1)_r} \frac{1+t}{(1-t)^{\lambda+2r-1}} = \lambda(1-t)^{\alpha-\lambda},$$

that is

$$\sum_{r=0}^{\infty} \frac{(\alpha+1)_{2r}}{r!\,(\alpha+1)_r} \frac{t^r}{1-t)^{2r}} = \frac{(1-t)^{\alpha+1}}{1+t} . \tag{6}$$

The steps are reversible so that (6) is equivalent to (1). If we put

$$z = t(1-t)^{-2},$$

then

$$1 - t = \frac{2}{1+\sqrt{1+4z}}, \quad \frac{1+t}{1-t} = \sqrt{1+4z}$$

and (6) becomes

$$\sum_{r=0}^{\infty} \frac{(a+1)_{2r} z^r}{r!\,(\alpha+1)_r} = \frac{1}{\sqrt{1+4z}} \left(\frac{1+\sqrt{1+4z}}{2} \right)^{-\alpha} \tag{7}$$

The formula (7) is well-known (see for example [1, p. 101, (6)]).

REFERENCES

[1] A. ERDÉLYI, W. MAGNUS, F. OBERHETTINGER and F. G. TRICOMI, Higher transcendental functions, vol. 1, New York, 1953.