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Summation of a special ${}_4F_3$

by L. CARLITZ (a Durham, U. S. A.) (*)

Summary. - *It is shown that*

$${}_4F_3 \left[\begin{matrix} -n, \frac{1}{2}(\alpha+1), \frac{1}{2}\alpha+1, \lambda+n; \\ \alpha+1, \frac{1}{2}(\lambda+1), \frac{1}{2}\lambda+1 \end{matrix} \right] = \frac{\lambda(\lambda-\alpha)_n}{(\lambda+2n)(\lambda)_n},$$

1. Consider the sum

$$\begin{aligned} A_n(\alpha, \lambda) &= \sum_{r=0}^n \frac{(-n)_r (\alpha+1)_{2r} (\lambda+n)_r}{r! (\alpha+1)_r (\lambda+1)_{2r}} \\ &= {}_4F_3 \left[\begin{matrix} -n, \frac{1}{2}(\alpha+1), \frac{1}{2}\alpha+1, \lambda+n; \\ \alpha+1, \frac{1}{2}(\lambda+1), \frac{1}{2}\lambda+1 \end{matrix} \right], \end{aligned}$$

which incidentally is Saalschützian. We shall show that

$$(1) \quad A_n(\alpha, \lambda) = \frac{\lambda(\lambda-\alpha)_n}{(\lambda+2n)(\lambda)_n}.$$

If we put

$$f_n(\alpha, \lambda) = \sum_{r=0}^n (-1)^r \binom{n}{r} (\alpha+r+1)_r (\lambda+2r+1)_{n-r-1} (\lambda+2n),$$

where it is understood that the term on the right corresponding to $r=n$ is

$$(-1)^n (\alpha+n+1)_n,$$

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then it is easily verified that (1) is equivalent to

$$(2) \quad f_n(\alpha, \lambda) = (\lambda - \alpha)_n.$$

Now we have

$$\begin{aligned} & f_n(\alpha + 1, \lambda) - f_n(\alpha, \lambda) \\ &= \sum_{r=0}^n (-1)^r \binom{n}{r} \{ (\alpha + r + 2)_r - (\alpha + r + 1)_r \} \cdot (\lambda + 2r + 1)_{n-r-1} (\lambda + 2n) \\ &= \sum_{r=0}^n (-1)^r \binom{n}{r} r (\alpha + r + 2)_{r-1} (\lambda + 2r + 1)_{n-r-1} (\lambda + 2n) \\ &= -n \sum_{r=0}^{n-1} (-1)^r \binom{n-1}{r} (\alpha + r + 3)_r (\lambda + 2r + 3)_{n-r-2} (\lambda + 2n) \end{aligned}$$

which evidently implies

$$(3) \quad f_n(\alpha + 1, \lambda) - f_n(\alpha, \lambda) = -n f_{n-1}(\alpha + 2, \lambda + 2).$$

Clearly (2) holds when $n = 0$. Assuming the truth of (2) for all values less than n , it follows from (3) that

$$(4) \quad f_n(\alpha + 1, \lambda) - f_n(\alpha, \lambda) = -n(\lambda - \alpha)_{n-1}.$$

If we keep λ fixed and note that $f_n(\alpha, \lambda)$ is a polynomial in α , it is clear from (4) that

$$(5) \quad f_n(\alpha, \lambda) = (\lambda - \alpha)_n + g_n(\lambda),$$

where $g_n(\lambda)$ is independent of α . We now take $\alpha = \lambda$, so that (5) becomes

$$f_n(\lambda, \lambda) = g_n(\lambda) \quad (n > 0).$$

But

$$\begin{aligned} f_n(\lambda, \lambda) &= \frac{(\lambda + 2n)(\lambda)_n}{\lambda} A_n(\lambda, \lambda) \\ &= \frac{(\lambda + 2n)(\lambda)_n}{\lambda} \sum_{r=0}^n \frac{(-n)_r (\lambda + n)_r}{r! (\lambda + 1)_r} \end{aligned}$$

and

$$\sum_{r=0}^n \frac{(-n)_r (\lambda + n)_r}{r! (\lambda + 1)_r} = \frac{(1 - n)_n}{(\lambda + 1)_n} = 0$$

for $n > 0$. Hence $g_n(\lambda) = 0$ and (5) reduces to (2).

2. We have

$$\begin{aligned} & \sum_{r=0}^{\infty} \frac{(\lambda + 2n)(\lambda)_n}{n!} A_n(\alpha, \lambda) t^n \\ &= \sum_{n=0}^{\infty} \frac{(\lambda + 2n)(\lambda)_n}{n!} t^n \sum_{r=0}^{\infty} (-1)^r \binom{n}{r} \frac{(\alpha + 1)_{2r} (\lambda + n)_r}{(\alpha + 1)_r (\lambda + 1)_{2r}} \\ &= \sum_{r=0}^{\infty} \frac{(\alpha + 1)_{2r} t^r}{r! (\alpha + 1)_r (\lambda + 1)_{2r}} \sum_{n=r}^{\infty} \frac{t^{n-r}}{(n-r)!} (\lambda + 2n)(\lambda)_{n+r} \\ &= \sum_{r=0}^{\infty} \frac{(\alpha + 1)_{2r} (\lambda)_{2r} t^r}{r! (\alpha + 1)_r (\lambda + 1)_{2r}} \sum_{n=0}^{\infty} \frac{t^n}{n!} (\lambda + 2r + 2n)(\lambda + 2r)_n \\ &= \sum_{r=0}^{\infty} \frac{(\alpha + 1)_{2r} t^r}{r! (\alpha + 1)_r (\lambda + 2r)} \left\{ (\lambda + 2r) \sum_{n=0}^{\infty} \frac{(\lambda + 2r)_n}{n!} t^n \right. \\ & \quad \left. + 2t(\lambda + 2r) \sum_{n=0}^{\infty} \frac{(\lambda + 2r + 1)_n}{n!} t^n \right\} \end{aligned}$$

$$= \lambda \sum_{r=0}^{\infty} \frac{(\alpha+1)_{2r} t^r}{r! (\alpha+1)_r} \{ (1-t)^{-\lambda-2r} + 2t(1-t)^{-\lambda-2r-1} \}.$$

It accordingly follows from (1) or (2) that

$$\lambda \sum_{n=0}^{\infty} \frac{(\alpha+1)_{2n} t^n}{n! (\alpha+1)_n} \frac{1+t}{(1-t)^{\lambda+2n-1}} = \lambda(1-t)^{\alpha-\lambda},$$

that is

$$(6) \quad \sum_{r=0}^{\infty} \frac{(\alpha+1)_{2r}}{r! (\alpha+1)_r} \frac{t^r}{1-t)^{2r}} = \frac{(1-t)^{\alpha+1}}{1+t}.$$

The steps are reversible so that (6) is equivalent to (1).

If we put

$$z = t(1-t)^{-2},$$

then

$$1-t = \frac{2}{1 + \sqrt{1+4z}}, \quad \frac{1+t}{1-t} = \sqrt{1+4z}$$

and (6) becomes

$$(7) \quad \sum_{r=0}^{\infty} \frac{(\alpha+1)_{2r} z^r}{r! (\alpha+1)_r} = \frac{1}{\sqrt{1+4z}} \left(\frac{1 + \sqrt{1+4z}}{2} \right)^{-\alpha}$$

The formula (7) is well-known (see for example [1, p. 101, (6)]).

REFERENCES

- [1] A. ERDÉLYI, W. MAGNUS, F. OBERHETTINGER and F. G. TRICOMI, Higher transcendental functions, vol. 1, New York, 1953.