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SEZIONE SCIENTIFICA

BREVI NOTE

A bilinear generating function for the Jacobi polynomials

by L. Carlitz (a Durham, U. S. A.) (*) (**)

Summary. - *A closed expression is obtained for the series*

$$\sum_{n=0}^{\infty} \frac{n!}{(\gamma)_n} (x-1)^n (y-1)^n w^n P_n^{(\alpha-n,\, -\alpha-\gamma-n)}\left(\frac{x+1}{x-1}\right) \cdot P_n^{(\beta-n,\, -\beta-\gamma-n)}\left(\frac{y+1}{y-1}\right).$$

Weisner [3] has proved the formula

(1) $$(1-w)^{\alpha+\beta-\gamma}(1+(x-1)w)^{-\alpha}(1+(y-1)w)^{-\beta}F(\alpha,\ \beta;\ \gamma;\ \zeta)$$

$$\sum_{n=0}^{\infty} \frac{(\gamma)_n w^n}{n!} F(-n,\alpha;\ \gamma;\ x) F(-n,\ \beta;\ \gamma;\ y),$$

where

$$\zeta = \frac{xyw}{(1+(x-1)w)(1+(y-1)w)}.$$

Since the Jacobi polynomial

(2) $$P_n^{(\alpha,\beta)}(x) = \sum_{r=0}^{n} \binom{\alpha+n}{n-r}\binom{\beta+n}{r}\left(\frac{x-1}{2}\right)^r\left(\frac{x+1}{2}\right)^{n+r}$$

$$= \frac{(\beta+1)n}{n!}\left(\frac{x-1}{2}\right)^n F\left[-n,\ -\alpha-n;\ \beta+1;\ \frac{x+1}{x-1}\right],$$

(1) can also be written in the following form:

$$\sum_{n=0}^{\infty} \frac{n!}{(r+1)_n} \frac{4^n w^n}{(x-1)^n (y-1)^n} P_n^{(\alpha-n,\gamma)}\left(\frac{x+1}{x-1}\right)$$

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$$\cdot P_n^{(\beta-n,\,\gamma)}\left(\frac{y+1}{y-1}\right)$$

$$=(1-w)^{-\alpha-\beta-\gamma-1}(1+(x-1)w)^{\alpha}(1+(y-1)w)^{\beta}$$

$$\cdot F(-\alpha,\ -\beta;\ \gamma+1;\ \zeta),$$

where ζ has the same meaning as above.

In view of (3) it would be of interest to find a bilinear generating function for the modified JACOBI polynomial

$$P_n^{(\alpha-n,\,\beta-n)}(x).$$

Indeed such a generating function is implied by (3) if we make use of the relation [2, p. 63].

$$P_n^{(\alpha-n,\,-\alpha-\gamma-n)}(x)=\left(\frac{1+x}{2}\right)^n P_n^{(\alpha-n,\,\gamma-1)}\left(\frac{3-x}{1+x}\right). \tag{4}$$

Incidentally (4) can be proved rapidly as follows: It follows easily from (2) that

$$\sum_{n=0}^{\infty} P_n^{(\alpha-n,\,\beta-n)}(x)t^n=\left(1+\frac{x+1}{2}\,t\right)^{\alpha}\left(1+\frac{x-1}{2}\right)^{\beta}$$

and [1, p. 120]

$$\sum_{n=0}^{\infty} P_n^{(\alpha-n,\,\beta)}(x)t^n=(1+t)^{\alpha}\left(1-\frac{x-1}{2}\,t\right)^{-\alpha-\beta-1}$$

$$\sum_{n=0}^{\infty}\left(\frac{1+x}{2}\right)^n t^n P_n^{(\alpha-n,\,\gamma-1)}\left(\frac{3-x}{1+x}\right)$$

$$=\left(1+\frac{x+1}{2}\,t\right)^{\alpha}\left(1+\frac{x-1}{2}\,t\right)^{-\alpha-\gamma},$$

$$=\sum_{n=0}^{\infty} P_n^{(\alpha-n,\,-\alpha-\gamma-n)}(x)t^n$$

and (4) follows at once.

If we now substitute from (4) in (3) we find that

$$\sum_{n=0}^{\infty} \frac{n!}{(\gamma)_n} (x-1)^n (y-1)^n w^n P_n^{(\alpha-n,\, -\alpha-\gamma-n)}\left(\frac{x+1}{x-1}\right) \cdot P_n^{(\beta-n,\, -\beta-\gamma-n)}\left(\frac{y+1}{y-1}\right) \tag{5}$$

$$= (1-w)^{-\alpha-\beta-\gamma}(1-xw)^{\alpha}(1-yw)^{\beta} F\left[-\alpha, -\beta; \gamma; \frac{(x-1)(y-1)w}{(1-xw)(1-yw)}\right].$$

In particular if $\gamma = -\alpha-\beta$, (5) reduces to

$$\sum_{n=0}^{\infty} \frac{n!}{(-\alpha-\beta)_n} (x-1)^n (y-1)^n w^n P_n^{(\alpha-n,\, \beta-n)}\left(\frac{x+1}{x-1}\right) \cdot P_n^{(\beta-n,\, \alpha-n)}\left(\frac{y+1}{y-1}\right) \tag{6}$$

$$= (1-xw)^{\alpha}(1-yw)^{\beta} F\left[-\alpha, -\beta; -\alpha-\beta; \frac{(x-1)(y-1)w}{(1-xw)(1-yw)}\right].$$

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