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A note on Bessel polynomials

Nota di S. K. CHATTERJEA (a Calcutta, India) (*)

Summary. - *In proving* GROSSWALD'S *theorem* CIMA *states that the result* $y_{2n}(x) > 0$, $x \leq -1$, *follows from the result* $y_{2n}(x)$ *has a real zero* x_0 (*if any*) $> -1/4$ *Here an independent proof is given.*

KRALL and FRINK [1] considered a system of polynomials $y_n(x)$, $(n = 0, 1, 2, ...)$, known as the BESSEL polynomials, where $y_n(x)$ is defined as the polynomial solution

(1) $$y_n(x) = \sum_{\nu=0}^{n} \frac{(n+\nu)!}{(n-\nu)!\,\nu!} \cdot \left(\frac{x}{2}\right)^{\nu}.$$

of the differential equation

(2) $$x^2 y'' + 2(x+1)y' - n(n+1)y = 0.$$

These polynomials satisfy the recurrence relation

(3) $$y_{n+1}(x) = (2n+1)xy_n(x) + y_{n-1}(x).$$

In a paper [2] E. GROSSWALD proved the following theorem:

(4) - For even n, the BESSEL polynomial $y_n(x)$ has no real zero. His proof mainly depends upon the following facts:

(5) - All coefficients of $y_{2n}(x)$ are positive and therefore any real zero of $y_{2n}(x)$ has to be negative;

(6) Any real zero x_0 of $y_{2n}(x)$ satisfies $x_0 > -\frac{1}{4}$;

(*) Pervenuta alla Segreteria dell' U. M. I. il 27 giugno 1962.

(7) $$y_{2n}(x) \geq (1+x)^{2n}/(2n)! \qquad \text{for} \qquad -\frac{1}{4} < x < 0.$$

Recently J. A. CIMA [3] remarked that the proof of (7) by GROSSWALD is incorrect, however the theorem (4) is true. The proof of (4) by CIMA depends upon the following facts:

(8) $$y_{2n}(x) > 0 \qquad \text{for} \qquad x \geq 0,$$

which follows from the fact that all coefficients of the BESSEL polynomials are positive;

(9) $$y_{2n}(x) > 0 \qquad \text{for} \qquad x \leq -1,$$

which follows from (6);

(10) $$y_{2n}(x) > 0 \qquad \text{for} \qquad -1 < x < 0,$$

which is proved by induction. Thus it is quite clear from the scheme of CIMA's proof that we need not prove $y_{2n}(x) > 0$ for $x \leq -\frac{1}{4}$. It is sufficient to prove thal $y_{2n}(x) > 0$ for $x \leq -1$.

The object of this note is show that the inequality (9) can be proved readily by means of the representation

(11) $$y_{2n}(x) y_{2n+2}(x)$$

$$= y_{2n}^2(x) + (4n+3)x^2 \sum_{k=1}^{2n} (2k+1) y_k^2(x) + (4n+3)x(x+1).$$

First we notice that

$$y_{2n}(x) y_{2n+2}(x) = y_{2n}^2(x) + (4n+3) x y_{2n}(x) y_{2n+1}(x)$$

Now we have [4]

$$y_n(x) y_{n+1}(x) = 1 + x \sum_{k=0}^{n} (2k+1) y_k^2(x)$$

Thus it follows that

$$y_{2n}(x)y_{2n+2}(x)$$

$$= y_{2n}^2(x) + (4n + 3)x^2 \sum_{k=0}^{2n} (2k + 1)y_k^2(x) + (4n + 3)x$$

$$= y_{2n}^2(x) + (4n + 3)x^2 \sum_{k=1}^{2n} (2k + 1)y_k^2(x) + (4n + 3)x(x + 1),$$

which is (11).

Now to prove (9) we first observe that for $n = 1$, $y_{2n}(x) = y_2(x) = 1 + 3x + 3x^2 = 1 + 3x(x + 1) > 0$ for $x \leq -1$. Next assume that $y_{2n}(x) > 0$ for $x \leq -1$. Then from (11) we at once observe that the right member is positive, since $x \leq -1$; again $y_{2n}(x) > 0$ for $x \leq -1$. Consequently $y_{2n+2}(x) > 0$. This completes the proof of (9).

Lastly from (11) we deduce the following inequalities:

(12) $$y_{2n+2}(x) > y_{2n}(x); \quad \text{for} \quad x \leq -1,\ x > 0.$$

(13) $$y_{2n+2}(x) > \{ 1 + (4n + 1)(4n + 3)x^2 \} y_{2n}(x); \ x \leq -1,\ x > 0.$$

(14) $$y_{2n}(x)y_{2n+2}(x) > (4n + 3)x^2 \sum_{k=1}^{2n} (2k + 1)y_k^2(x); \ x \leq -1,\ x > 0$$

(15) $$y_{2n}(x)y_{2n+2}(x) > y_{2n}^2(x) + (4n + 3)x^2 \sum_{k=1}^{2n} (2k+1)y_k^2(x); \ x \leq -1,\ x > 0.$$

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