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SEZIONE SCIENTIFICA

BREVI NOTE

The summability by logarithmic means of the ultraspherical series.

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Summary. - *In this paper the author studies the summability by Riesz logarithmic means of order one for ultraspherical series.*

1. Let $f(\theta, \varphi)$ be a function defined for the range $0 \leq \theta \leq \pi$, $0 \leq \varphi \leq 2\pi$ on a sphere S. The ultraspherical series associated with this function is

$$(1.1) \qquad f(\theta, \varphi) \sim \frac{1}{2\pi} \sum_{n=0}^{\infty} (n + \lambda) \iint_S \frac{P_n^{(\lambda)}(\cos \omega) f(\theta', \varphi') d\sigma'}{[\sin^2 \theta' \sin^2 (\varphi - \varphi')]^{\frac{1}{2} - \lambda}}, \quad \lambda > 0,$$

where

$$\cos \omega = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos (\varphi - \varphi'),$$

and $P_n^{(\lambda)}(x)$ is defined as follows:

$$(1.2) \qquad (1 - 2\rho\mu + \mu^2)^{-\lambda} = \sum_{n=0}^{\infty} \rho^n P_n^{(\lambda)}(\mu), \qquad \lambda > 0.$$

We suppose throughout that the function

$$(1.3) \qquad f(\theta', \varphi'([\sin^2 \theta' \sin^2 (\varphi - \varphi')]^{\lambda - \frac{1}{2}}$$

is integrable (L) over the whole surface of the unit sphere.

The generalised mean value of $f(\theta, \varphi)$ is defined as follows:

$$(1.4) \qquad f(\omega) = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2} + \lambda\right)}{\Gamma(\lambda) \cdot 2\pi(\sin \omega)^{2\lambda}} \int_{C_\omega} \frac{f(\theta', \varphi') ds'}{[\sin^2 \theta' \sin^2 (\varphi - \varphi')]^{\frac{1}{2} - \lambda}},$$

(*) Pervenuta alla Segreteria dell' U. M. I. il 4 Settembre 1961.

where the integral is taken along the small circle whose centre is (θ, φ) on the sphere and whose curvilinear radius is ω.

We write

$$s_m(\omega) = \sum_{k=0}^{m} (k + \lambda) P_k^{(\lambda)}(\cos \omega)$$

$$L_n(\omega) = \frac{1}{\log n} \sum_{m=1}^{n} \frac{s_m(\omega)}{m} \sin \omega$$

and

$$\Phi(\omega) = \frac{\Gamma(\lambda)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}+\lambda\right)} f(\omega) \cdot (\sin \omega)^{2\lambda-1}.$$

DEFINITION. – The sequence of partial sums $\{S_n\}$ is said to be summable by RIESZ logarithmic means of order one or summable $(R, \log n, 1)$ if

$$\lim \frac{1}{\log n} \sum_{k=1}^{n} \frac{S_k}{k}$$

exists.

The object of this paper is to prove the following theorem:

THEOREM. – If $0 < \lambda < 1$ and

$$\int_t^{\pi} \frac{|\Phi(u)|}{u^{1+\lambda}} du = o\left(\log \frac{1}{t}\right), \qquad \text{as } t \to 0,$$

then the series (1.1) is summable-$(R, \log n, 1)$ to the value zero.

For $\lambda = \frac{1}{2}$, we get a corresponding result for LAPLACE series.

2. We require the following lemmas:

LEMMA 1. – If, as $t \to 0$,

$$\int_t^{\pi} \frac{|\Phi(u)|}{u^{1+\lambda}} du = o\left(\log \frac{1}{t}\right)$$

then

$$\int_0^{t} \frac{|\Phi(u)|}{u^{\lambda}} du = o\left(t \log \frac{1}{t}\right) \tag{2.1}$$

PROOF. - We have

$$\int_0^t \frac{|\Phi(t)|}{t^\lambda}\,dt = -\left[\left\{-t\int_t^\pi \frac{|\Phi(t)|}{t^{1+\lambda}}\,dt\right\}_0^t + \int_0^t\left\{\int_t^\pi \frac{|\Phi(t)|}{t^{1+\lambda}}\,dt\right\}dt\right]$$

$$= o\left(t\log\frac{1}{t}\right).$$

LEMMA 2. - If $o \leq \omega \leq \frac{1}{n}$ and $0 < \lambda < 1$, then

$$\text{(2.2)}\qquad L_n(\omega) = O\left(\frac{n^{2\lambda+1}\cdot\omega}{\log n}\right).$$

PROOF. - From SZEGÖ [3] we have

$$P_n^{(\lambda)}(\cos\omega) = O(n^{2\lambda-1}) \quad \text{for} \quad 0 \leq \omega \leq \pi,$$

consequently

$$s_m(\omega) = O\left\{\sum_{k=0}^{m}(k+\lambda)\cdot k^{2\lambda-1}\right\}$$

$$= O(m^{2\lambda+1}).$$

It follows that

$$L_n(\omega) = \frac{1}{\log n}\sum_{m=1}^{n}\frac{O(m^{2\lambda+1})\cdot\omega}{m}$$

$$= O\left(\frac{n^{2\lambda+1}\cdot\omega}{\log n}\right).$$

LEMMA 3. - For $\frac{1}{n} \leq \omega \leq \pi - \frac{1}{n}$

$$L_n(\omega) = O\left[\frac{1}{\log n}\cdot\frac{1}{\omega^\lambda\left(\sin\frac{\omega}{2}\right)^\lambda}\right] + O\left[\frac{1}{\log n}\cdot\frac{1}{\omega(\sin\omega)^\lambda}\right]$$

$$+ O\left[\frac{1}{\log n}(\sin\omega)^{\lambda+\varepsilon}\right] + O\left[\frac{1}{\log n}\,\frac{1}{\omega^\lambda(\sin\omega)^{1-\varepsilon}}\right]$$

where $\varepsilon > 0$ such that $\lambda + \varepsilon < 1$.

PROOF. - We have [[1], Lemmas 8 and 9]

$$s_m(\omega)\sin(\omega) = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}+\lambda\right)}{\Gamma(\lambda)}$$

$$\left[R\left\{\omega^{-\lambda}\cdot\omega^{\lambda}\left(\cot\frac{\omega}{2}\right)^{1-\lambda}\left(\sin\frac{\omega}{2}\right)^{1-2\lambda}\left(m+\lambda+\frac{1}{2}\right)^{\lambda} e^{\frac{i\pi\lambda}{2}}\cdot e^{i\left(m+\lambda+\frac{1}{2}\right)\omega}\right\}\right]$$

$$+\,O\left[m^{\lambda-1}\omega^{-1}(\sin\omega)^{-\lambda}+m^{-1}(\sin\omega)^{-\lambda-1}+m^{\lambda-1}\omega^{-\lambda}(\sin\omega)^{-1}\right].$$

Therefore

$$\sum_{m=1}^{n}\frac{s_m(\omega)\sin\omega}{m} = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}+\lambda\right)}{\Gamma(\lambda)}\left(\cot\frac{\omega}{2}\right)^{1-\lambda}\left(\sin\frac{\omega}{2}\right)^{1-2\lambda}\left[\left\{\cos\left\{\left(\lambda+\frac{1}{2}\right)\omega+\frac{\lambda\pi}{2}\right\}\right.\right.$$

$$\left.\sum_{m=1}^{n}\left(1+\frac{\lambda+\frac{1}{2}}{m}\right)^{\lambda}\frac{\cos m\omega}{m^{1-\lambda}}\right\}-\left\{\sin\left\{\left(\lambda+\frac{1}{2}\right)\omega+\frac{\lambda\pi}{2}\right\}\right.$$

$$\left.\left.\sum_{m=1}^{n}\left(1+\frac{\lambda+\frac{1}{2}}{m}\right)^{\lambda}\cdot\frac{\sin m\omega}{m^{1-\lambda}}\right\}\right]$$

$$+\sum_{m=1}^{n}\frac{1}{m^{2-\lambda}(\sin\omega)^{\lambda}\cdot\omega}+\sum_{m=1}^{n}\frac{1}{m^{1-\varepsilon}}\cdot\frac{1}{(m\sin\omega)^{1-\varepsilon}}\cdot\frac{1}{(\sin\omega)^{\lambda+\varepsilon}}$$

$$+\sum_{m=1}^{n}\frac{1}{m^{2-\lambda-\varepsilon}}\cdot\frac{1}{(m\sin\omega)^{\varepsilon}}\cdot\frac{1}{\omega^{\lambda}(\sin\omega)^{1-\varepsilon}}$$

$$= O\left[\frac{1}{\left(\sin\frac{\omega}{2}\right)^{\lambda}}\right]+\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}+\lambda\right)}{\Gamma(\lambda)}\left(\cot\frac{\omega}{2}\right)^{1-\lambda}\left(\sin\frac{\omega}{2}\right)^{1-2\lambda}$$

$$\left[\left\{\cos\left\{\left(\lambda+\frac{1}{2}\right)\omega+\frac{\lambda\pi}{2}\right\}\right.\right.$$

$$\sum_{m=2}^{n}\left\{1+\frac{\lambda\left(\lambda+\frac{1}{2}\right)}{m}+\frac{\lambda(\lambda-1)}{L^2}\left(\frac{\lambda+\frac{1}{2}}{m}\right)^{2}+\dots\right\}\frac{\cos m\omega}{m^{1-\lambda}}\right\}$$

$$-\left\{\sin\left\{\left(\lambda+\frac{1}{2}\right)\omega+\lambda\frac{\pi}{2}\right\}\sum_{m=2}^{n}\left\{1+\frac{\lambda\left(\lambda+\frac{1}{2}\right)}{m}+\ldots\right\}\frac{\sin m\omega}{m^{1-\lambda}}\right\}\Bigg]$$

$$+O\left[\frac{1}{\omega(\sin\omega)^{\lambda}}\right]+O\left[\frac{1}{(\sin\omega)^{\lambda+\varepsilon}}\right]+O\left[\frac{1}{\omega^{\lambda}(\sin\omega)^{1-\varepsilon}}\right]$$

Therefore

$$L_n(\omega)=O\left[\frac{1}{\log n}\cdot\frac{1}{\omega^{\lambda}\left(\sin\frac{\omega}{2}\right)^{\lambda}}\right]+O\left[\frac{1}{\log n}\cdot\frac{1}{\omega(\sin\omega)^{\lambda}}\right]$$

$$+O\left[\frac{1}{\log n}\cdot\frac{1}{(\sin\omega)^{\lambda+\varepsilon}}\right]+O\left[\frac{1}{\log n}\cdot\frac{1}{\omega^{\lambda}(\sin\omega)^{1-\varepsilon}}\right]$$

Lemma 4. - For $\frac{1}{n}\leq\omega\leq\frac{\pi}{2}$, we have

$$L_n(\omega)=O\left[\frac{1}{\log n}\cdot\frac{1}{\omega^{1+\lambda}}\right]+O\left[\frac{1}{\log n}\cdot\frac{1}{\omega^{\lambda+\varepsilon}}\right]+O\left[\frac{1}{\log n}\cdot\frac{1}{\omega^{\lambda+1-\varepsilon}}\right].$$

The proof follows from Lemma 3.

Lemma 5. - For $\pi-\frac{1}{n}\leq\omega\leq\pi$ and $0<\lambda<1$ we have

$$L_n(\omega)=O\left[\frac{n^{\lambda}}{\log n}\right]$$

Proof. - We have [[3], putting $\delta=0$ in equation 2.2,]

$$(\sin\omega)^{\lambda}\,|\,s_m(\omega)\,|\leq\frac{C\cdot A^{\lambda}{}_{m}}{\sin\frac{\omega}{2}}$$

$$\sum_{m=1}^{n}\frac{s_m(\omega)}{m}\sin\omega\leq\sum_{m=1}^{n}\frac{C\cdot(\sin\omega)^{1-\lambda}A^{\lambda}{}_{m}}{m\cdot\sin\frac{\omega}{2}}$$

$$=O(n^{\lambda})$$

Thus

$$L_n(\omega) = O\left(\frac{n^\lambda}{\log n}\right)$$

3. Proof of the theorem. -

The m^{th} partial sum of the series (1.1) is given by

$$S_m - f(P) = \frac{\Gamma(\lambda)}{2\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}+\lambda\right)} \int_0^\pi \{ f(\omega) - f(0) \} s_m(\omega) \cdot (\sin \omega)^{2\lambda} \cdot d\omega.$$

Without loss of generality we may set

$$f(P) = f(0) = 0$$

consequently

$$S_m = \int_0^\pi \Phi(\omega) s_m(\omega) \sin \omega d\omega.$$

Let t_n be the first logarithmic mean of the series (1.1), then

$$t_n = \frac{1}{\log n} \sum_{m=1}^{n} \int_0^\pi \Phi(\omega) \frac{s_m(\omega)}{m} \cdot \sin \omega d\omega$$

$$= \frac{1}{\log n} \int_0^\pi \Phi(\omega) \cdot \sum_{m=1}^{n} \frac{s_m(\omega)}{m} \cdot \sin \omega d\omega$$

$$= \int_0^\pi \Phi(\omega) L_n(\omega) d\omega.$$

$$= \left[\int_0^{\frac{1}{n}} + \int_{\frac{1}{n}}^{\frac{\pi}{2}} + \int_{\frac{\pi}{2}}^{\pi - \frac{1}{n}} + \int_{\pi - \frac{1}{n}}^{\pi} \right] \Phi(\omega) L_n(\omega) d\omega$$

$$= L_1 + L_2 + L_3 + L_4, \text{ say.}$$

Now

$$L_1=\int_0^{\frac{1}{n}} \Phi(\omega)L_n(\omega)d\,\omega$$

$$=\int_0^{\frac{1}{n}} \frac{|\Phi(\omega)|}{\omega^\lambda}\cdot\omega^\lambda\cdot O\left(\frac{n^{2\lambda+1}\cdot\omega}{\log n}\right)d\omega$$

(3.1) $$=0(1)\quad\text{as}\quad n\to\infty\quad\text{by Lemma 1.}$$

$$L_2=\int_{\frac{1}{n}}^{\frac{\pi}{2}} |\Phi(\omega)|\cdot O\left(\frac{1}{\log n}\cdot\frac{1}{\omega^{1+\lambda}}\right)d\omega+o(1)$$

$$=O\left(\frac{1}{\log n}\right)\int_{\frac{1}{n}}^{\frac{\pi}{2}}\frac{|\Phi(\omega)|}{\omega^{1+\lambda}}d\omega+o(1)$$

(3.2) $$=0(1)\quad\text{as}\quad n\to\infty.$$

$$L_3=O\left(\frac{1}{\log n}\right)\Bigg[\int_{\frac{\pi}{2}}^{\pi-\frac{1}{n}}\frac{|\Phi(\omega)|}{\omega^\lambda\cdot\left(\sin\frac{\omega}{2}\right)^\lambda}d\omega+\int_{\frac{\pi}{2}}^{\pi-\frac{1}{n}}\frac{|\Phi(\omega)|}{\omega\cdot(\sin\omega)^\lambda}d\omega$$

$$+\int_{\frac{\pi}{2}}^{\pi-\frac{1}{n}}\frac{|\Phi(\omega)|}{(\sin\omega)^{\lambda+\varepsilon}}d\omega+\int_{\frac{\pi}{2}}^{\pi-\frac{1}{n}}\frac{|\Phi(\omega)|}{\omega^\lambda\cdot(\sin\omega)^{1-\varepsilon}}d\omega\Bigg]$$

(3.3) $$=0(1)\quad\text{as}\quad n\to\infty.$$

And

$$L_4 = \int_{\pi - \frac{1}{n}}^{\pi} |\Phi(\omega)| \cdot O\left(\frac{n^\lambda}{\log n}\right) \cdot d\omega$$

(3.4) $\qquad = 0(1) \quad \text{as} \quad n \to \infty.$

Combining (3.1), (3.2), (3.3(and (3.4) the theorem is proved.

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