Bollettino
Unione Matematica Italiana

Richard Bellman

Some questions concerning difference approximations to partial differential equations.

Bollettino dell'Unione Matematica Italiana, Serie 3, Vol. **17** (1962), n.2, p. 188–190.

Zanichelli

<http://www.bdim.eu/item?id=BUMI_1962_3_17_2_188_0>

Some questions concerning difference approximations to partial differential equations

Nota di RICHARD BELLMAN (*)

Summary. - *It is shown that difference approximations of unconventional type can be extremely useful in analytic and computational studies of partial differential equations, particularly in conserving boundedness and nonnegativity properties.*

1. Introduction

The study of finite-difference approximations to partial differential equations is well advanced, and the increasing use and significance of digital computers guarantees that this study will continue to be actively pursued. One question, however, that does not seem to have been discussed to any extent is that of finding discrete approximations that manifestly exhibit the boundedness, positivity, and so on, of the original continuous operator.

Thus, for example, the recurrence relation

$$u(t + \Delta) = (1 - a\Delta)u(t), \quad u(0) = c, \tag{1.1}$$

yields the nonnegativity we associate with the solution of

$$u' = - au, \quad u(0) = c, \tag{1.2}$$

provided that $a\Delta < 1$. On the other hand, the relation

$$u(x, t + \Delta^2) = \frac{u(x + \Delta\sqrt{2}, t) + u(x - \Delta\sqrt{2}, t)}{2}, \tag{1.3}$$

$t = 0, \Delta^2, \dots,$ shows very clearly the nonnegativity associated with the solution of

$$u_t = u_{xx}, \quad u(x, 0) = g(x). \tag{1.4}$$

Approximations of this type are very useful computationally because of their stability properties. We shall discuss briefly

(*) Pervenuta alla Segreteria dell'U. M. I. il 18 maggio 1962.

below the problem of obtaining approximations of arbitrarily high degree with the same property, and the related problem for

$$u_t = uu_x + g(u). \tag{1.5}$$

For previous work and computational results, see [1].

2. The one-dimensional heat equation

To obtain higher-order approximations possessing the property that nonnegativity is preserved, we can proceed as follows. Write

$$u(x,\ t + \Delta^2) = \sum_{i=1}^{N} w_i(u(x + r_i\Delta,\ t) + u(x - r_i\Delta,\ t)) \tag{2.1}$$

Using the relations $u_t = u_{xx}$, $u_{tt} = {}_{xxxx}$, and those of higher degree, we obtain a set of moment relations

$$\begin{aligned} 1 &= \sum_i w_i\,, \\ \frac{1}{2} &= \sum_i w_i r_i^2, \end{aligned} \tag{2.2}$$

and so on. The question of existence of real r_i and positive w_i yielding an error which is $0(\Delta^{2k})$ for a given k and N can then be determined by invoking classical moment theory.

More interesting would be a general result to the effect that any linear partial differential equation with constant coefficients can be approximated by difference relations of arbitrarily high degree which preserve the nonnegativity and boundedness of the original equation. Let us further note that a nonlinear approximation, e. g., one arising from a branching process, may be superior in that it requires fewer terms to obtain an equivalent approximation.

3. The equation $u_t = uu_x$

Consider now the equation

$$u_t = uu_x,\ u(x,\ 0) = g(u), \tag{3.1}$$

which is often used (see [1]) as a simple model of an equation generating a shock wave. Since it possesses an explicit analytic solution, it is useful in testing numerical integration techniques. An approximation to terms in $0(\Delta^2)$ is obtained using the difference relation

$$u(x,\ t+\Delta) = u(x + u(x,\ t)\Delta,\ t), \tag{3.2}$$

$t = 0,\ \Delta, \ldots$.

An approximation to terms in $0(\Delta^2)$ is obtained from

$$u(x,\ t+\Delta) = u(x + u(x + u(x,\ t)\Delta,\ t)\Delta,\ t). \tag{3.3}$$

One might suspect that arbitrarily accurate approximations could be obtained by continuing in this fashion, but it seems difficult to establish this.

Similarly, the equation

$$u_t = uu_x + g(u) \tag{3.4}$$

has the approximation

$$u(x,\ t+\Delta) = u(x + u(x,\ t)\Delta,\ t) + g(u)\Delta, \tag{3.5}$$

valid to $0(\Delta^2)$, and

$$\begin{aligned} u(x,\ t+\Delta) &= u(x + u(x + u(x,\ t)\Delta,\ t)\Delta + \frac{g(u)\Delta^2}{2},\ t) \\ &\quad + g(u(x + u(x,\ t)\Delta)\Delta + \frac{g'(u)g(u)\Delta^2}{2}. \end{aligned} \tag{3.6}$$

valid to $0(\Delta^3)$.

REFERENCE

[1] Bellman R., I. Cherry, and G. M. Wing, *A Note on the Numerical Integration of a Class of Nonlinear Hyperbolic Equations*, « Quarterly of Applied Mathematics », Vol. 16, 1958, pp. 181-183.