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On a generating function of Laguerre polynomials

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Summary. - *A generating function for the Laguerre polynomials is derived from their recurtion formulae.*

Introduction: The following generating function of LAGUERRE polynomials $L_n^{(\alpha)}(x)$.:

$$\sum_{n=0}^{\infty} \frac{L_n^{(\alpha)}(x) t^n}{\Gamma(n+\alpha+1)} = e^t (xt)^{-\frac{1}{2}\alpha} J_\alpha[2(xt)^{\frac{1}{2}}]; \qquad (\alpha > -1) \tag{1}$$

is due to DOETSCH and follows from

$$\sum_{n=0}^{\infty} L_n^{(\alpha)}(x) t^n = (1-t)^{-\alpha-1} exp\left(\frac{xt}{t-1}\right); \qquad (|t| < 1) \tag{2}$$

by means of the LAPLACE transformation. Recently TSCHAUNER [1] proved the known generating function for GEGENBAUER polynomials $C_n^{p+\frac{1}{2}}(x)$, $n = 0, 1, 2, \ldots; \; p > -\frac{1}{2}$; :

$$\sum_{n=0}^{\infty} \frac{C_n^{p+\frac{1}{2}}(x)}{\Gamma(1+n+2p)} t^n = \frac{\Gamma(1+p)}{\Gamma(1+2p)} e^{xt} \left(\frac{yt}{2}\right)^{-p} J_p(yt), \tag{3}$$

where $y = (1-x^2)^{\frac{1}{2}}$
from the recursion formula. Following the method adopted by TSCHAUNER we like to prove (1) from the recursion formula of LAGUERRE polynomials:

$$(n+2)u_{n+2} - (2n+3+\alpha-x)u_{n+1} + (n+\alpha+1)u_n = 0 \tag{4}$$

where $u_n \equiv u_n(x, \alpha) = L_n^{(\alpha)}(x)$, $u_0 = 1$, $u_1 = \alpha + 1 - x$.

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Now using

$$v_n(x, \alpha) = \frac{u_n(x, \alpha)}{\Gamma(n + \alpha + 1)} = \frac{L_n^{(\alpha)}(x)}{\Gamma(n + \alpha + 1)}$$

we derive from (4) the recursion formula for $v_n(x, \alpha)$:

(5) $$(n + 2)(n + \alpha + 2)v_{n+2} - (2n + 3 + \alpha - x)v_{n+1} + v_n = 0$$

with

$$v_0 = \frac{1}{\Gamma(\alpha + 1)}, \qquad v_1 = \frac{\alpha + 1 - x}{(\alpha + 1)\Gamma(\alpha + 1)},$$

so that $(\alpha + 1)v_1 - (\alpha + 1 - x)v_0 = 0$.

Next multiplying both members of (5) by t^n and summing from $n = 0$ to ∞ we obtain the following homogeneous differential equation of order two:

(6) $$t\ddot{V} + (\alpha + 1 - 2t)\dot{V} - (\alpha + 1 - x - t)V = (\alpha + 1)v_1 - (\alpha + 1 - x)v_0 = 0,$$

where

$$V(t) = \sum_{n=0}^{\infty} v_n t^n.$$

Using the substitution

$$V(t) = e^t (t)^{-\frac{1}{2}\alpha} W(t)$$

we derive from (6) the following differential equation

(7) $$4t^2\ddot{W} + 4t\dot{W} + (4tx - \alpha^2)W = 0$$

Finally using the substitution

$$\sqrt{t} = z \text{ and } 2\sqrt{x} = y$$

we obtain from (7) the Bessel differential equation

(8) $$z^2 \frac{d^2 W}{dz} + z\frac{dW}{dz} + (y^2 z^2 - \alpha^2)W = 0,$$

which leads to the particular solutions

$$W = AJ_\alpha(yz), \qquad W = BY_\alpha(yz).$$

Thus the particular solutions of (6) are

$$(9) \qquad V(t) = Ae^t(t)^{-\frac{1}{2}\alpha} J_\alpha(2\sqrt{xt}), \qquad V(t) = Be^t(t)^{-\frac{1}{2}\alpha} Y_\alpha(2\sqrt{xt}).$$

Now since $V(0) = v_0$ is finite, we take the particular solution

$$V(t) = Ae^t t^{-\frac{1}{2}\alpha} J_\alpha(2\sqrt{xt}).$$

To determine the normalising factor A, we notice

$$V(0) = v_0 = \frac{1}{\Gamma(\alpha+1)} = A \cdot \frac{(2\sqrt{x})^\alpha}{2^\alpha \Gamma(\alpha+1)},$$

whence $A = x^{-\frac{1}{2}\alpha}$.

Thus we obtain the generating function for the LAGUERRE polynomials

$$(10) \qquad \begin{aligned} V(t) &= \sum_{n=0}^{\infty} v_n(x, \alpha)t^n = \sum_{n=0}^{\infty} \frac{L_n^{(\alpha)}(x)t^n}{\Gamma(n+\alpha+1)} \\ &= e^t(tx)^{-\frac{1}{2}\alpha} J_\alpha(2\sqrt{xt}); \qquad (\alpha > -1). \end{aligned}$$

In particular, when $\alpha = 0$, we derive from (10)

$$(11) \qquad \sum_{n=0}^{\infty} \frac{L_n(x)t^n}{n!} = e^t J_0(2\sqrt{xt}).$$

Next using $\alpha = -\frac{1}{2}$, $x = x^2$ and noticing the relation

$$H_{2n}(x) = (-)^n 2^{2n} n! L_n^{\left(-\frac{1}{2}\right)}(x^2)$$

we derive ultimately

$$(12) \qquad \sum_{n=0}^{\infty} \frac{(-)^n H_{2n}(x) z^{2n}}{2^{2n}(2n)!} = e^{\frac{z^2}{4}} \cos(zx),$$

where $z = 2\sqrt{t}$.

Again using $\alpha = \frac{1}{2}$, $x = x^2$ and noticing the relation

$$H_{2n+1}(x) = (-)^n 2^{2n+1} n!\, x L_n^{\left(\frac{1}{2}\right)}(x^2)$$

we similarly derive

$$\sum_{n=0}^{\infty} \frac{(-)^n H_{2n+1}(x) z^{2n+1}}{2^{2n+1}(2n+1)!} = e^{\frac{z^2}{4}} \sin (zx). \tag{13}$$

Lastly from (12) and (13) we obtain on taking $z = 2$:

$$\sum_{n=0}^{\infty} \frac{(-)^n H_{2n}(x)}{(2n)!} = e \cos (2x) \tag{14}$$

$$\sum_{n=0}^{\infty} \frac{(-)^n H_{2n+1}(x)}{(2n+1)!} = e \sin (2x); \tag{15}$$

both of which may be compared with the formulae in [2].

REFERENCES

[1] J. TSCHAUNER, *Zur Erzeugung der Gegenbauerschen Polynome*, « Math Zeitschr. », Band 75, Seite 1-2, (1961).

[2] W. MAGNUS & F. OBERHETTINGER, *Formulas and theorems for the functions of mathematical physics*, Translated by J. Wermer, New York, N. Y., (1954), p. 81.