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Some q-identities related to the theta functions

Nota di LEONARD CARLITZ (a Durham, U. S. A.) (*) (1)

Summary. - *The identies (8), (22), (23) are obtained.*

1. It is familiar that if

$$F(x)=\prod_{0}^{\infty}(1-q^{2n-1}x)(1-q^{2n-1}x^{-1})=\sum_{-\infty}^{\infty}A_n x^n$$

where $|q|<1$, then

(1) $$F(x)=-xqF(q^2x),$$

so that

$$A_n=-q^{2n-1}A_{n-1}=(-1)^n q^{n^2}A_0 .$$

Thus

$$F(x)=A_0\sum_{-\infty}^{\infty}q^{n^2}x^n.$$

If we put

(2) $$(F(x))^{-1}=\sum_{-\infty}^{\infty}C_n x^n$$

and apply the same method we apparently get

$$q^{2n-1}C_n=-C_{n-1}, \qquad C_n=(-1)^n q^{-n^2}.$$

It follows that

$$(F(x))^{-1}=\sum_{-\infty}^{\infty}(-1)^n q^{-n^2}x^n .$$

This result is obviously false, the right member never converges.

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The explanation of this curious situation is that the right member of (2) converges only in the ring

$$|q| < |x| < |q|^{-1}$$

and it is therefore not permissible to apply (1).

We may however derive a valid expansion for $(F(x))^{-1}$ in the following way. Making use of the familiar identity

$$\prod_{0}^{\infty}(1-q^{2n}x)^{-1} = \sum_{0}^{\infty}\frac{x^n}{(n)!},$$

where

(3) $$(n)! = (1-q^2)\dots(1-q^{2n}), \qquad (0)! = 1,$$

we get

$$(F(x))^{-1} = \sum_{r=0}^{\infty}\frac{q^r x^r}{(r)!}\sum_{s=0}^{\infty}\frac{q^s x^{-s}}{(s)!}$$

$$= \sum_{n=-\infty}^{\infty} x^n \sum_{r-s=n}\frac{q^{r+s}}{(r)!\,(s)!}.$$

It follows that

(4) $$(F(x))^{-1} = 1 + \sum_{n=1}^{\infty} q^n(x^n + x^{-n})\sum_{s=0}^{\infty}\frac{q^{2s}}{(s)!\,(n+s)!}.$$

Now on the other hand it is known (see for example [5, p. 489, ex. 14]) that

(5) $$\frac{Kk^{\frac{1}{2}}}{\pi}\frac{\theta_4(0,\ q)}{\theta_4(z,\ q)} = \frac{1}{2}a_0 + \sum_{1}^{\infty} a_n \cos 2nz,$$

where

(6) $$a_n = 2\sum_{r=0}^{\infty}(-1)^r q^{\left(r+\frac{1}{2}\right)\left(2n+r+\frac{1}{2}\right)} = 2q^{\frac{1}{4}}\sum_{r=0}^{\infty}(-1)^r q^{r(r+1)+n(2r+1)}.$$

Since

$$\frac{Kk^{\frac{1}{2}}}{\pi}\theta_4(0,\ q) = q^{\frac{1}{4}}\prod_{1}^{\infty}(1-q^{2n})^3,$$

(5) becomes

$$q^{\frac{1}{4}} \prod_{1}^{\infty} \frac{(1-q^{2n})^2}{(1-q^{2n-1}e^{2iz})(1-q^{2-1}e^{-2iz})} = \frac{1}{2}a_0 + \frac{1}{2}\sum_{1}^{\infty} a_n(e^{2niz}+e^{-2niz}).$$

Replacing e^{2iz} by x we get

$$(7) \qquad q^{\frac{1}{4}} \prod_{1}^{\infty} \frac{(1-q^{2n})^2}{(1-q^{2n-1}x)(1-q^{2n-1}x^{-1})} = \frac{1}{1}a_0 + \sum_{1}^{\infty} a_n(x^n+x^{-n}).$$

Comparing (7) with (4) and using (6) we get

$$(8) \qquad \sum_{s=0}^{\infty} \frac{q^{2s}}{(s)!\,(n+s)!} \prod_{r=1}^{\infty} (1-q^{2r})^2 = \sum_{r=0}^{\infty} (-1)^r q^{r(r+1)+2nr} \qquad (n \geq 0).$$

We shall now give a direct proof of (8).

2. Put

$$(9) \qquad \prod_{0}^{\infty} (1-q^{2n}xt)(1-q^{2n}x^{-1}t) = \sum_{-\infty}^{\infty} (-1)^n x^n I_n(t, q^2),$$

$$(10) \qquad \prod_{1}^{\infty} (1-q^2_n xt)^{-1}(1-q^{2n}x^{-1}t)^{-1} = \sum_{-\infty}^{\infty} x^n \bar{I}_n(t, q^2).$$

The functions $I_n(t)$, $\bar{I}_n(t)$ were introduced (in different notation by JACKSON [3, 4] and have been discussed recently by HAHN [2]; see also [1]. It follows from (9) and (10) that

$$(11) \qquad I_n(t, q^2) = \sum_{r=0}^{\infty} \frac{q^{r(r-1)+(n-r)(n-r-1)}\,t^{n+2r}}{(r)!\,(Mn+r)!},$$

$$(12) \qquad \bar{I}_n(t, q^2) = \sum_{r=1}^{\infty} \frac{t^{n+2r}}{(r)!\,(n+r)!},$$

where for convenience we put

$$\frac{1}{(-n)!} = 0 \qquad (n = 1, 2, 3, \ldots).$$

Using (11) it is not difficult to verify that

$$\sum_{-\infty}^{\infty} q^{-n(n-1)} x^n I_n(q^2 t, q^2) = \prod_{1}^{\infty} \frac{1-q^{2n+2}t^2}{(1-q^{2n+2}xt)(1-q^{2n}x^{-1}t)},$$

which implies

$$I_n(qt,\ q^2) = q^{n^2}\bar{I}_n(t,\ q^2) \prod_1^{\infty} (1 - q^{2r}t^2). \tag{13}$$

In particular, for $t = q$, (13) becomes

$$I_n(q^2,\ q^2) = q^{n^2} Q\, \bar{I}_n(q,\ q^2), \tag{14}$$

where

$$Q = \prod_1^{\infty} (1 - q^{2r}). \tag{15}$$

In JACOBI's identity

$$Q \prod_1^{\infty} (1 - q^{2n-1}x)(1 - q^{2n-1}) = \sum_{-\infty}^{\infty} (-1)^n q^n x^n, \tag{16}$$

replace x by qx. Then

$$Q \prod_1^{\infty} (1 - q^{2n}x)(1 - q^{2n-2}x^{-1}) = \sum_{-\infty}^{\infty} (-1)^n q^{n^2+n} x^n,$$

so that

$$Q \prod_1^{\infty} (1 - q^{2n+2}x)(1 - q^{2n+2}x^{-1}) = (1 - x^{-1})^{-1} \sum_{-\infty}^{\infty} (-1)^n q^{n^2+n} x^n.$$

Comparing this with (9) we get

$$Q\, I_n(q^2,\ q^2) = q^{n(n+1)} \sum_{r=0}^{\infty} (-1)^r q^{r(r+1)+2nr}. \tag{17}$$

Thus (14) becomes

$$q^{-n} Q^2 \bar{I}_n(q,\ q^2) = \sum_{r=0}^{\infty} (-1)^r q^{r(r+1)+2nr}. \tag{18}$$

But by (12)

$$\bar{I}_n(q,\ q^2) = \sum_{r=0}^{\infty} \frac{q^{n+2r}}{(r)!\,(n+r)!},$$

so that (18) is identical with (8),

3. It is evident from (9) and (10) that

$$I_{-n}(t, q^2) = I_n(t, q^2), \qquad \bar{I}_{-n}(t, q^2) = \bar{I}_n(t, q^2).$$

We show that if

(19) $$\sigma_n = \sum_{r=0}^{\infty} (-1)^r q^{r(r+1)+n(2r+1)},$$

then

(20) $$\sigma_{-n} = \sigma_n .$$

It follows that (8) holds for all integral n.

To prove (20) we observe that

$$\sigma_{-n} = \sum_{r=0}^{\infty} (-1)^r q^{r(r+1)+n(2r+1)} = \sum_{r=0}^{2n-1} (-1)^r q^{r(r+1)-n(2r+1)}$$

$$+ \sum_{r=0}^{\infty} (-1)^r q^{r(r+1)+n(2r+1)} .$$

But

$$\sum_{r=0}^{2n-1} (-1)^r q^{r(r+1)-n(2r+1)} = q^{-n} \sum_{r=0}^{2n-1} (-1)^r q^{r(r+1-2n)}$$

$$= q^{-n} \sum_{r=0}^{2n-1} (-1)^{r+1} q^{(2n-1-r)(-r)}$$

$$= -q^{-n} \sum_{r=0}^{2n-1} (-1)^r q^{(r+1-2n)}$$

and (20) follows at once.

It is clear from (6) and (19) that (5) can be written in the following way.

$$\frac{Q^3}{\theta_4(z, q)} = \sigma_0 + \sum \sigma_n (e^{2niz} + e^{-2niz}).$$

In view of (20), this becomes

(21) $$\frac{Q^3}{\theta_4(z, q)} = \sum_{-\infty}^{\infty} \sigma_n e^{2niz} .$$

4. If we replace q^2 by q, (8) becomes

$$\sum_{s=0}^{\infty} \frac{q^s}{(q)_s(q^{n+1})} \prod_{r=1}^{\infty} (1-q^r)(1-q^{n+r}) = \sum_{r=0}^{\infty} (-1)^r q^{\frac{1}{2}r(r+1)+nr}.$$

where

$$(a)_s = (1-a)(1-qa)\dots(1-q^{s-1}a), \qquad (a)_0 = 1.$$

This suggestes the following more general identity:

$$\sum_{s=0}^{\infty} \frac{q^s}{(q)_s(a)_s} \prod_{r=1}^{\infty} (1-q^r)(1-q^{r-1}a) = \sum_{r=0}^{\infty} (-1)^r q^{\frac{1}{2}r(r-1)} a^r. \tag{22}$$

This identity is indeed true and can be proved rapidly as follows. We have

$$\sum_{s=0}^{\infty} \frac{q^s}{(q)_s(a)_s} \prod_{r=0}^{\infty} (1-q^r a) = \sum_{s=0}^{\infty} \frac{q^s}{(q)_s} \prod_{r=0}^{\infty} (1-q^{r+s}a)$$

$$= \sum_{s=0}^{\infty} \frac{q^s}{(q)_s} \sum_{n=0}^{\infty} (-1)^n \frac{q^{\frac{1}{2}n(n-1)}(q^s a)^n}{(q)_n}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{q^{\frac{1}{2}n(n-1)} a^n}{(q)_n} \sum_{s=0}^{\infty} \frac{q^{(n+1)s}}{(q)_s}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{q^{\frac{1}{2}n(n-1)} a^n}{(q)_n} \prod_{r=1}^{\infty} (1-q^{n+r})^{-1}$$

$$= \sum_{n=0}^{\infty} (-1)^n q^{\frac{1}{2}n(n-1)} a^n \prod_{r=1}^{\infty} (1-q^r)^{-1}.$$

This evidently proves (22).

The identity

$$\sum_{s=0}^{\infty} \frac{(b)_s x^s}{(q)_s(a)_s} \prod_{r=0}^{\infty} \frac{(1-q^r a)(1-q^r x)}{1-q^r bx} \tag{23}$$

$$= \sum_{n=0}^{\infty} (-1)^n q^{\frac{1}{2}n(n-1)} \frac{(x)_n a^n}{(q)_n(bx)_n},$$

reduces to (22) for $b=0$, $x=a$.

To prove (23) we take

$$\sum_{s=0}^{\infty} \frac{(b)_s x^s}{(q)_s (a)_s} \prod_{r=0}^{\infty} (1 - q^r a) = \sum_{s=0}^{\infty} \frac{(b)_s x^s}{(q)_s} \prod_{r=0}^{\infty} (1 - q^{r+s} a)$$

$$= \sum_{s=0}^{\infty} \frac{(b)_s x^s}{(q)_s} \sum_{n=0}^{\infty} (-1)^n \frac{q^{\frac{1}{2}n(n-1)} (q^s a)^n}{(q)_n}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{q^{\frac{1}{2}n(n-1)} a^n}{(q)_n} \sum_{s=0}^{\infty} \frac{(b)_s}{(q)_s} (q^n x)^s$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{q^{\frac{1}{2}n(n-1)} a^n}{(q)_n} \prod_{r=0}^{\infty} \frac{1 - q^{n+r} bx}{1 - q^{n+r} bx}$$

$$= \prod_{r=0}^{\infty} \frac{1 - q^r bx}{1 - q^r x} \sum_{n=0}^{\infty} (-1)^n q^{\frac{1}{2}n(n-1)} \frac{(x)_n a^n}{(q)_n (bx)_n} .$$

This proves (23).

When $b = 0$, (23) becomes

$$\sum_{s=0}^{\infty} \frac{x^s}{(q)_s (a)_s} \prod_{s=0}^{\infty} (1 - q^r a)(1 - q^r x) = \sum_{r=0}^{\infty} (-1)^r q^{\frac{1}{2}r(r-1)} \frac{(x)_r}{(q)_r} a^r .$$

In particular, for $a = q^{n+1}$, (24) implies

(25) $$Q \prod_{r=0}^{\infty} (1 - q^{2r} x^2) \bar{I}_n(x, q^2) = x^n \sum_{r=0}^{\infty} (-1)^r q^{r(r+1)+2nr} \frac{1}{(r)!} \prod_{s=0}^{r-1} (1 - q^{2s} x^2.$$

which reduces to (18) when $x = q$.

REFERENCES

[1] L. Carlitz, *Some formulas related to the Rogers-Ramanujan identities*, « Annali di Matematica pura ed applicata», (4), vol. 47 (1959), pag. 243-251.

[2] W. Hahn, *Beiträge zur Theorie der Heineschen Reihen*, « Mathematische Nachrichten », vol. 2 (1949), pp. 340-379.

[3] F. H. Jackson, *The application of basic numbers to Bessel's and Legendre's functions* (first paper), « Proceedings of the London Mathematical Society », (2), vol. 2 (1905), pp. 192-220.

[4] F. H. Jackson, *The application of basic numbers to Bessel's and Legendre's functions* (second paper), « Proceedings of the London Mathematical Society », (2), vol. 3 (1905), pp. 1-23.

[5] E. T. Whittaker and G. N. Watson, *Modern analysis*, 4 th edition, Cambrigde, 1927.