Bollettino Unione Matematica Italiana

G. S. Pandey

An aspect of local property for the convergence of Jacobi series.

Bollettino dell'Unione Matematica Italiana, Serie 3, Vol. **17** (1962), n.1, p. 98–107.

Zanichelli

<http://www.bdim.eu/item?id=BUMI_1962_3_17_1_98_0>

An aspect of local property for the convergence of Jacobi series.

By G. S. PANDEY (a Chhatarpur, India). (*)

Summary. - *In the present paper the author establishes a condition under which the convergence of Jacobi series at an internal point of the interval* $(-1, +1)$ *be comes a local property.*

1. Let $f(x)$ be a function defined by $-1 \leq x \leq 1$ and such that the integral $\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta}f(x)dx$ exists. The JACOBI series corresponding to the function $f(x)$ is

$$f(x) \sim \sum_{n=0}^{\infty} a_n P_n^{(\alpha,\beta)}(x), \tag{1.1}$$

where

$$a_n = \frac{\Gamma(n+1)\Gamma(n+\alpha+\beta+1)}{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)} \cdot \frac{2n+\alpha+\beta+1}{2^{\alpha+\beta+1}} \cdot \int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta}P_n^{(\alpha,\beta)}(x)\cdot f(x)dx; \tag{1.2}$$

and the JACOBI polynomials $P_n^{(\alpha,\beta)}(x)$, $\alpha > -1$, $\beta > -1$ are defined by the following expansion:

$$\frac{1}{\sqrt{1-2xt+t^2}}(1-t+\sqrt{1-2xt+t^2})^{-\alpha}(1+t+\sqrt{1-2xt+t^2})^{-\beta} = \sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x)\cdot t^n.$$

The ultraspherical series

$$\sum_{n=0}^{\infty} b_n P_n^{(\lambda)}(x), \tag{1.3}$$

Pervenuta alla Segreteria dell'U. M. I. il 15 marzo 1962.

is the particular case of the series (1.1) for the value $\alpha = \beta = \lambda - \frac{1}{2}$ and the Legendre series

$$(1.4) \qquad \sum_{n=0}^{\infty} c_n P_n(x),$$

is the particular case of the series (1.3) for the value $\frac{1}{2}$ of the parameter λ.

Under the condition $c_n = o(n^{1/2})$, YOUNG (3) proved that the convergence of the series (1.4) becomes a local property.

D. P. GUPTA (1) has extended recently, the result of YOUNG to the ultraspherical series and he has shown that:
If

$$b_n = o(n^{1-\lambda}),$$

then the convergence of the series (1.3) at an internal point x of the interval $(-1, +1)$ depends only on the behaviour of the generating function $f(x)$ in the immediate neighbourhood of the point.

The object of this paper is to find condition such that the convergence of the series (1.1) is a local property at any point of the interval $(-1, +1)$.

We are led to prove the following result:

THEOREM: *If*

$$(1.5) \qquad a_n = o(n^{1/2}),$$

then the convergence of the series (1.1) *at an internal point* x *of the interval* $(-1, +1)$ *depends only on the behaviour of the generating function* f(x) *in the immediate neighbourhood of the point.*

2. The proof of the theorem requires the help of the following lemmas:

LEMMA 1 (2, p. 194). – If α and β be arbitrary real numbers. Then

$$(2.1) \qquad P_n^{(\alpha,\beta)}(\cos\theta) = \frac{\cos\left[\{n + (\alpha+\beta+1)/2\}\theta - \left(\alpha + \frac{1}{2}\right)\frac{\pi}{2}\right]}{n^{1/2}\pi^{1/2}\left(\sin\frac{\theta}{2}\right)^{\alpha+1/2}\left(\cos\frac{\theta}{2}\right)^{\beta+1/2}} + O(n^{-3/2})$$

$$0 < \theta < \pi.$$

LEMMA 2 (2, p. 167):

$$P_n^{(\alpha,\beta)}(\cos\theta) = \theta^{-\alpha-1/2}\cdot O(n^{-1/2}),$$

(2.2)

$$0 < \theta \leq \frac{\pi}{2},\ \alpha \geq -\frac{1}{2};$$

$$P_n^{(\alpha,\beta)}(\cos\theta) = O(n^{-1/2}),$$

(2.3)

$$0 < \theta \leq \frac{\pi}{2},\ \alpha \leq -\frac{1}{2}.$$

LEMMA 3 (2, p. 71):

$$\sum_{\nu=0}^{n} \frac{2\nu+\alpha+\beta+1}{2^{\alpha+\beta+1}}\cdot\frac{\Gamma(\nu+1)\Gamma(\nu+\alpha+\beta+1)}{\Gamma(\nu+\alpha+1)\Gamma(\nu+\beta+1)}\cdot P_\nu^{(\alpha,\beta)}(t)P_\nu^{(\alpha,\beta)}(x)$$

(2.4)
$$= \frac{2^{-\alpha-\beta}}{(2n+\alpha+\beta+2)}\cdot\frac{\Gamma(n+2)\Gamma(n+\alpha+\beta+2)}{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}$$

$$\cdot\frac{P_{n+1}^{(\alpha,\beta)}(t)P_n^{(\alpha,\beta)}(x) - P_n^{(\alpha,\beta)}(t)P_{n+1}^{(\alpha,\beta)}(x)}{t-x}.$$

3. *Proof of the theorem.* The n th partial sum of the series (1.1) is,

$$S_n(x) = \sum_{\nu=0}^{n}\frac{\Gamma(\nu+1)\Gamma(\nu+\alpha+\beta+1)}{\Gamma(\nu+\alpha+1)\Gamma(\nu+\beta+1)}\cdot\frac{2\nu+\alpha+\beta+1}{2^{\alpha+\beta+1}}$$

$$\cdot\int_{-1}^{1}(1-t)^\beta(1+t)^\alpha P_n^{(\alpha,\beta)}(t)P_n^{(\alpha,\beta)}(x)f(t)dt.$$

Using lemma 3, i. e., (2.4), we have

$$S_n(x) = \frac{2^{-\alpha-\beta}}{(2n+\alpha+\beta+2)}\cdot\frac{\Gamma(n+2)\Gamma(n+\alpha+\beta+2)}{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}$$

(3.1)
$$\int_{-1}^{1}\frac{P_{n+1}^{(\alpha,\beta)}(t)P_n^{(\alpha,\beta)}(x)-P_n^{(\alpha,\beta)}(t)P_{n+1}^{(\alpha,\beta)}(x)}{t-x}(1-t)^{\alpha}(1+t)^{\beta}f(t)\,dt$$
$$=K_n\left[\int_{-1}^{-1+\varepsilon}+\int_{-1+\varepsilon}^{x-\varepsilon'}+\int_{x-\varepsilon'}^{x+\varepsilon'}+\int_{x+\varepsilon'}^{1-\varepsilon}+\int_{1-\varepsilon}^{1}\right]$$
$$=J_1+J_2+J_3+J_4+J_5,$$

say, having taken $\varepsilon<x+1-\varepsilon'$, $1-x-\varepsilon'$ and

$$K_n=\frac{2^{-\alpha-\beta}}{(2n+\alpha+\beta+2)}\cdot\frac{\Gamma(n+2)\Gamma(n+\alpha+\beta+2)}{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}\sim O(n).$$

Now,

(3.2)
$$J_2=K_n\int_{-1+\varepsilon}^{x-\varepsilon'}(1-t)^{\alpha}(1+t)^{\beta}\cdot\frac{P_{n+1}^{(\alpha,\beta)}(t)P_n^{(\alpha,\beta)}(x)-P_n^{(\alpha,\beta)}(t)P_{n+1}^{(\alpha,\beta)}(x)}{t-x}f(t)\,dt$$
$$=J_{2,1}-J_{2,2},\ \text{say}.$$

Putting $t=\cos\theta$ and denoting

$$\left|\{n+1+(\alpha+\beta+1)/2\}\theta-\left(\alpha+\frac{1}{2}\right)\frac{\pi}{2}\right|$$ by ω_{n+1}, we have

by lemma 1,

$$J_{2,1}=K_n\int_{-1+\varepsilon}^{x-\varepsilon'}(1-t)^{\alpha}(1+t)^{\beta}\cdot\frac{f(t)}{t-x}\cdot P_{n+1}^{(\alpha,\beta)}(t)P_n^{(\alpha,\beta)}(x)\,dt$$

$$=K_n\int_{-1+\varepsilon}^{x-\varepsilon'}(1-\cos\theta)^{\alpha}(1+\cos\theta)^{\beta}\cdot\frac{f(t)}{n-x}\cdot P_n^{(\alpha,\beta)}(x)$$

$$\cdot\left[\frac{\cos\omega_{n+1}}{(n+1)^{1/2}\pi^{1/2}\left(\sin\frac{\theta}{2}\right)^{\alpha+1/2}\left(\cos\frac{\theta}{2}\right)^{\beta+1/2}}+O(n^{-3/2})\right]\cdot dt$$

$$= K_n \int_{-1+\varepsilon}^{x-\varepsilon'} 2^{\alpha+\beta} \left(\sin\frac{\theta}{2}\right)^{2\alpha} \left(\cos\frac{\theta}{2}\right)^{2\beta} \cdot \frac{f(t)}{t-x} P_n^{(\alpha,\beta)}(x)$$

$$\cdot \left[\frac{\cos \omega_{n+1}}{(n+1)^{1/2}\pi^{1/2}\left(\sin\frac{\theta}{2}\right)^{\alpha+1/2}\left(\cos\frac{\theta}{2}\right)^{\beta+1/2}} + O(n^{-3/2}\right] dt$$

$$= K_n \int_{-1+\varepsilon}^{x-\varepsilon'} \left(\sin\frac{\theta}{2}\right)^{\alpha-1/2} \left(\cos\frac{\theta}{2}\right)^{\beta-1/2} \cdot 2^{\alpha+\beta} \cdot \frac{\cos\omega_{n+1}}{(n+1)^{1/2}\,\pi^{1/2}} \cdot \frac{f(t)}{t-x}\, dt$$

$$+ K_n \int_{-1+\varepsilon}^{x-\varepsilon'} \left(\sin\frac{\theta}{2}\right)^{2\alpha} \left(\cos\frac{\theta}{2}\right)^{2\beta} \cdot 2^{\alpha+\beta} \cdot P_n^{(\alpha,\beta)}(x)\, O(n^{-3/2}) \frac{f(t)}{t-x}\, dt$$

$$= O(n)\left| P_n^{(\alpha,\beta)}(x)\right| (n+1)^{-1/2} \int_{-1+\varepsilon}^{x-\varepsilon'} \left|\left(\sin\frac{\theta}{2}\right)^{\alpha-1/2}\left(\cos\frac{\theta}{2}\right)^{\beta-1/2}\right| \left|\cos\omega_{n+1}\right| \cdot \frac{|f(t)|}{|t-x|} dt$$

$$+ O(n)\left|P_n^{(\alpha,\beta)}(x)\right| \cdot O(n^{-3/2}) \int_{-1+\varepsilon}^{x-\varepsilon'} \left|\left(\sin\frac{\theta}{2}\right)^{2\alpha}\left(\cos\frac{\theta}{2}\right)^{2\beta} \cdot \right| \left|\frac{f(t)}{t-x}\right| dt$$

$$= O(n^{1/2} P_n^{(\alpha,\beta)}) \int_{-1+\varepsilon}^{x-\varepsilon'} \left|\left(\sin\frac{\theta}{2}\right)^{\alpha-1/2}\left(\cos\frac{\theta}{2}\right)^{\beta-1/2}\right| \cdot \left|\frac{f(t)}{t-x}\right| \left|\cos\omega_{n+1}\right| dt$$

$$+ O(n^{-1/2} \cdot P_n^{(\alpha,\beta)}(x)) \int_{-1+\varepsilon}^{x-\varepsilon'} \left|\left(\sin\frac{\theta}{2}\right)^{2\alpha}\left(\cos\frac{\theta}{2}\right)^{2\beta}\right| \cdot \left|\frac{f(t)}{t-x}\right| dt$$

$$= O(1) \int_{-1+\varepsilon}^{x-\varepsilon'} \left|\left(\sin\frac{\theta}{2}\right)^{\alpha-1/2}\left(\cos\frac{\theta}{2}\right)^{\beta-1/2}\right| \cdot \left|\cos\omega_{n+1}\right| \cdot \left|\frac{f(t)}{t-x}\right| dt$$

$$+ O(n^{-1}) \int_{-1+\varepsilon}^{x-\varepsilon'} \left(\sin\frac{\theta}{2}\right)^{2\alpha}\left(\cos\frac{\theta}{2}\right)^{2\beta} \left|\frac{f(t)}{t-x}\right| dt$$

$$= J_{2,1,1} + J_{2,1,2}, \text{ say.}$$

By RIEMANN-LEBESGUE theorem,

$$J_{2,1,1} = o(1). \tag{3.3}$$

Also,

$$J_{2,1,2} = O(n^{-1}) \int_{-2+\varepsilon}^{x-\varepsilon'} \left| \left(\sin\frac{\theta}{2}\right)^{2\alpha} \left(\cos\frac{\theta}{2}\right)^{2\beta} \right| \cdot \left| \frac{f(t)}{t-x} \right| dt$$

$$= o(1), \text{ as } n \to \infty. \tag{3.4}$$

Combining (3.3) and (3.4). We have

$$J_{2,1} = o(1). \tag{3.5}$$

Treating $J_{2,2}$ in the same manner as $J_{2,1}$, we find that

$$J_{2,2} = o(1). \tag{3.6}$$

Combining (3.5) and (3.6), we have

$$J_2 = o(1). \tag{3.7}$$

Now,

$$J_5 = K_n \int_{1-\varepsilon}^{1} (1-t)^{\alpha}(1+t)^{\beta} \frac{f(t)}{t-x} [P_{n+1}^{(\alpha,\beta)}(t) P_n^{(\alpha,\beta)}(x) -$$

$$P_n^{(\alpha,\beta)}(t) P_{n+1}^{(\alpha,\beta)}(x)] dt \tag{3.8}$$

$$= J_{5,1} - J_{5,2},$$

say.

Now,

$$J_{5,1} = K_n \int_{1-\varepsilon}^{1} (1-t)^{\alpha}(1+t)^{\beta} \frac{f(t)}{t-x} P_{n+1}^{(\alpha,\beta)}(t) P_n^{(\alpha,\beta)}(x) dt$$

$$= O(n) \mid P_n^{(\alpha,\beta)} \mid \int_{1-\varepsilon}^{1} (1-t)^{\alpha}(1-t)^{\beta} \mid P_{n+1}^{(\alpha,\beta)}(t) \mid \left| \frac{f(t)}{t-x} \right| dt.$$

Putting $t = \cos\varphi$, $1 - \varepsilon = \cos\eta$, we have, on the lines of SZEGÖ, (2, p. 253) keeping ε sufficiently small

$$J_{5,1} = O(n)O(n^{-1/2})\int_0^{\eta} \mid P_{n+1}^{(\alpha,\beta)} \mid \varphi^{2\alpha+1}(\cos\varphi) \mid f\cos\varphi) \mid d\varphi.$$

From (2.2) and (2.3) it follows that:

$$J_{5,1} = \begin{cases} O(1)\displaystyle\int_0^{\eta} \varphi^{-\alpha-1/2} \cdot \varphi^{2\alpha+1} \mid f(\cos\varphi) \mid d\varphi, & \text{if } \alpha \geq -\frac{1}{2}; \\ O(1)\displaystyle\int_0^{\eta} \varphi^{2\alpha+1} \mid f(\cos\varphi) \mid d\varphi, & \text{if } \alpha \leq -\frac{1}{2}. \end{cases}$$

(3.9)

$$= \begin{cases} O(1)\displaystyle\int_{-1}^{1} (1-t)^{\alpha/2-1/4}(1+t)^{\beta/2-1/4} \mid f(t) \mid dt, & \text{if } \alpha \geq -\frac{1}{2}; \\ O(1)\displaystyle\int_{-1}^{1} (1-t)^{\alpha}(1+t)^{\beta} \mid f(t) \mid dt, & \text{if } \alpha \leq -\frac{1}{2}. \end{cases}$$

But by hypothesis (1.5), we have

$$\frac{\Gamma(n+1)\Gamma(n+\alpha+\beta+1)}{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)} \cdot \frac{2n+\alpha+\beta+1}{2^{\alpha+\beta+1}}$$

$$\int_{-1}^{1} (1-t)^{\alpha}(1-t)^{\beta} P_n^{(\alpha,\beta)}(t)dt = o(n^{1/2})$$

i. e. $$O(n)\int_{-1}^{1} (1-t)^{\alpha}(1+t)^{\beta} \mid P_n^{(\alpha,\beta)}(t) \mid \; \mid f(t) \mid dt = o(n^{1/2})$$

or,

$$(3.10)\qquad O(n^{1/2})\int_{-1}^{1}(1-t)^{\alpha}(1+t)^{\beta}\mid P_n^{(\alpha,\beta)}(t)\mid\cdot\mid f(t)\mid dt=o(1).$$

The left hand member is

$$O(n^{1/2})\left[\int_{-1}^{-1+\xi}+\int_{-1+\xi}^{0}+\int_{0}^{1-\xi}+\int_{1-\xi}^{1}\right]$$

$$=I_1+I_2+I_3+I_4,$$

say, having taken $0<\xi<1$.

Now, using lemma 1, we have

$$I_3=O(n^{1/2})\int_{0}^{1-\xi}(1-t)^{\alpha}(1+t)^{\beta}\mid P_n^{(\alpha,\beta)}(t)\mid\ \mid f(t)\mid dt$$

$$=O(n^{1/2})\int_{\pi/2}^{\delta}(1-\cos\psi)^{\alpha}(1+\cos\psi)^{\beta}\mid P_n^{(\alpha,\beta)}(\cos\psi)\mid\ \mid f(\cos\psi)\mid d\psi$$

$$=O(n^{1/2})\int_{\pi/2}^{\delta}\left(\sin\frac{\psi}{2}\right)^{2\alpha}\left(\cos\frac{\psi}{2}\right)^{2\beta}\left[\frac{\cos\omega_n}{n^{1/2}\pi^{1/2}\left(\sin\frac{\psi}{2}\right)^{\alpha+1/2}\left(\cos\frac{\psi}{2}\right)^{\beta+1/2}}+\right.$$

$$\left.O(n^{-3/2})\right]\mid f(\cos\psi)\mid d\psi,$$

$$=I^{3,1}+I^{3,2},\ \text{say}$$

where $t=\cos\psi$, $1-\xi=\cos\delta$ and

$$\omega_n=\left[\{n+(\alpha+\beta+1)/2\}\theta-\left(\alpha+\frac{1}{2}\right)\frac{\pi}{2}\right].$$

Now,

$$I^{3,1}=O(1)\int_{\pi/2}^{\delta}\left(\sin\frac{\psi}{2}\right)^{\alpha-1/2}\left(\cos\frac{\psi}{2}\right)^{\beta-1/2}\cos\omega_n\mid f(\cos\psi)\mid d\psi$$

(3.11)
$$= o(1),$$

by RIEMANN-LEBESGUE theorem.

Also,

$$I_{3,1} = O(n^{1/2})O(n^{-3/2})\int_{\pi/2}^{\delta}\left(\sin\frac{\psi}{2}\right)^{2\alpha}\left(\cos\frac{\psi}{2}\right)^{1\beta} |f(\cos\psi)|\, d\psi$$

(3.12)
$$= o(1), \text{ as } n \to \infty.$$

Combining (3.11) and (3.12), we have

$$I_3 = o(1).$$

Similarly,

$$I_2 = o(1).$$

Again,

$$I_4 = O(n^{1/2})\int_{1-\xi}^{1}(1-t)^{\alpha}(1+t)^{\beta} \,|\, P_n^{(\alpha,\beta)}(t) \,|\, dt \cdot$$

$$= \begin{cases} O(1)\displaystyle\int_{-1}^{1}(1-t)^{\alpha/2-1/4}(1+t)^{\beta/2-1/4} |f(t)|\, dt, & \text{if } \alpha \geq -\frac{1}{2}; \\ O(1)\displaystyle\int_{-1}^{1}(1-t)^{\alpha}(1+t)^{\beta} |f(t)|\, dt, & \text{if } \alpha \leq -\frac{1}{2}. \quad 2, \text{ p. } 253). \end{cases}$$

We get similar expressions, for I_1 also,

But by hypothesis, we have

$$I_1 + I_2 + I_3 + I_4 = o(1).$$

Substituting the values of I_1, I_2, I_3 and I_4 as derived above, we have

$$O(1)\int_{-1}^{1}(1-t)^{\alpha/2-1/4}(1+t)^{\beta/2-1/4} |f(t)|\, dt = o(1),$$

$$\text{(3.13)} \qquad \text{if } \alpha \geq -\frac{1}{2}.$$

or,

$$\text{(3.14)} \qquad O(1)\int_{-1}^{1}(1-t)^{\alpha}(1+t)^{\beta}\,|\,f(t)\,|\,dt = (1), \qquad \text{if } \alpha \leq -\frac{1}{2}.$$

Using (3.13) and (3.14), we see from (3.9) that

$$\text{(3.15)} \qquad J_{5,1} = o(1).$$

$J_{5,2}$ can also be disposed off in exactly the same manner as $J_{5,1}$ and combining with (3.15), we have

$$\text{(3.16)} \qquad J_5 = o(1).$$

J_4 and J_1 can also be treated in exactly the same way as J_2 and J_5 respectively, and finally, therefore, we get

$$\text{(3.17)} \qquad \lim_{n\to\infty}\left[S_n(x) - K_n\int_{x-\varepsilon'}^{x+\varepsilon'}(1-t)^{\alpha}(1+t)^{\beta}\frac{f(t)}{t-x}\,|\,P_{n+1}^{(\alpha,\beta)}(t)P_n^{(\alpha,\beta)}(x) - P_n^{(\alpha,\beta)}(t)P_{n+1}^{(\alpha,\beta)}(x)\,|\,dt\right] = 0,$$

ε' being any positive quantity, arbitrarily small, but less than $x+1$ and $1-x$.

This completes the proof of the theorem.

I wish to express my warmest tanhs to Dr. D. P. GUPTA for much help and advice.

REFERENCES

[1] GUPTA D. P., *An aspect of local property for convergence of ultraspherical series*, «Iudian Journal of Maths., Vol. I», No. 2 (1959), p. 55-66.

[2] SZEGÖ G., «Orthogonal Polynomials, American Mathematical Society, Colloquium Publications», Vol. XXIII, 1959.

[3] YOUNG W. H., *On the connexion between Legendre series and Fourier series*, «P. L. M. S. (2), Vol. 18 (1919), p. 141-162.