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On convergenge of series of Jacobi polynomials

Nota By G. S. Pandey

Summary. - *In this paper the uniform convergence of the series of Jacobi polynomials has been investigated. The result obtained includes as a particular case that of Zitarosa (1948) for Legendre series.*

1. We consider the convergence of the series of Jacobi polinomials i. e. the series,

$$\sum_{n=0}^{\infty} a_n P_n^{(\alpha,\beta)}(x), \tag{1.1}$$

where
$P_n^{(\alpha,\beta)}$ is the coefficient of t^n in the expansion of

$$\frac{2}{\sqrt{1-2xt+t^2}}(1-t+\sqrt{1-2xt+t^2})^{-\alpha}(1+t+\sqrt{1-2xt+t^2})^{-\beta}.$$

This series is not necessarily a Fourier–Jacobi series corresponding to some function $f(x)$. This means that a_n need not be given by the formula

$$a_n = \frac{\Gamma(n+1)\Gamma(n+\alpha+\beta+1)}{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)} \cdot \frac{2n+\alpha+\beta}{2^{\alpha+\beta+1}} \cdot$$

$$\cdot \int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta} f(x) P_n^{(\alpha,\beta)}(x)dx.$$

The ultraspherical polynomials $P_n^{(\lambda)}(x)$ are the particular case of Jacobi polynomials for $\alpha=\beta=\lambda-\frac{1}{2}$.
We have infact the relation

$$P_n^{\left(\lambda-\frac{1}{2},\,\lambda-\frac{1}{2}\right)}(x) = A_n^{\left(\lambda-\frac{1}{2}\right)} P_n^{(\lambda)}(x),$$

(*) Pervenuta alla Segreteria dell'U. M. I. l' 8 novembre 1961.

where

$$A_n^{\left(\lambda-\frac{1}{2}\right)} = \binom{n+\lambda-\frac{1}{2}}{\lambda-\frac{1}{2}} \sim n^{\lambda-\frac{1}{2}}.$$

These ultraspherical polynomials $P_n^{(\lambda)}(x)$ reduce to LEGENDRE polynomials in case $\lambda = \frac{1}{2}$.

For the series

$$\frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos nx, \tag{1.2}$$

KOLMOGOROFF [See (5) p. 109] has proved the following theorem: If $a_n \to 0$, and the sequence $\{a_n\}$ is of bounded variation, the series (1.2) converges, save for $x = 0$, to an integrable function $f(x)$, and is the FOURIER series of $f(x)$.

For the series of LEGENDRE polynomials ZITAROSA [(4)] has proved an analogous theorem. Recently, D. P. GUPTA [(1)] has extended ZITAROSA's result to series of ultrasperical polynomials and he has shown that:

If

$$b_n = 0(n^{\mu}), \tag{1.3}$$

and the sequence

$$\left\{ \frac{b_n}{(n+\lambda)^{\mu}} \right\} \tag{1.4}$$

is of bounded variation for $0 < \mu < \lambda$ and $0 < \lambda \leq \frac{1}{2}$, then series

$$\sum_{n=0}^{\infty} b_n P_n^{(\lambda)}(x) \tag{1.5}$$

converges uniformly in the interval $(-1+\delta,\ 1-\delta)$.

The object of this paper is to study this problem of uniform convergence for the series (1.1).

We are led to prove the following result:

THEOREM. - If

$$a_n = 0(n^{\mu}), \tag{1.6}$$

and the sequence

$$\left\{ \frac{a_n}{(2n+\alpha+\beta+1)^{\mu}} \right\} \tag{1.7}$$

is of bounded variation for $0 < \mu < \frac{1}{2} - \alpha$, $0 < \alpha < \frac{1}{2}$; *then the series* (1.1) *is uniformly convergent in the interval* $(-1+\delta,\ 1-\delta)$

2. In the proof of the theorem we shall make use of the following lemmas:

LEMMA 1 (see 3, p. 71)

$$\sum_{\nu=0}^{n} \frac{(2\nu+\alpha+\beta+1)}{2^{\alpha+\beta+1}} \cdot \frac{\Gamma(\nu+\alpha+\beta+1)}{\Gamma(\alpha+1)\Gamma(\nu+\beta+1)} \cdot P_{\nu}^{(\alpha,\beta)}(x)$$

$$= \frac{(n+\alpha+1)}{2^{\alpha+\beta}(2n+\alpha+\beta+2)} \cdot \frac{\Gamma(n+\alpha+\nu+2)}{\Gamma(\alpha+1)\Gamma(n+\beta+1)} \cdot \tag{2.1}$$

$$\cdot \frac{P_n^{(\alpha,\beta)}(x) - \dfrac{n+1}{n+\alpha+1} P_{n+1}^{(\alpha,\beta)}(x)}{1-x}.$$

LEMMA 2 (see ,2)

$$\left| P_n^{(\alpha,\beta)}(\cos\theta) \right| < A(\varepsilon) n^{-\frac{1}{2}}, \tag{2.2}$$

where

$$0 < \alpha \leq \frac{1}{2}, -\frac{1}{2} < \beta, \ \varepsilon \leq \theta \leq \pi - \varepsilon,$$

and $A(\varepsilon)$ *denotes a constant for a fixed* $\varepsilon > 0$ *not necessarily the same at each occurrence.*

3. *Proof of the theorem:*

By lemma 1, we have

$$\sum_{\nu=0}^{n} \frac{2\nu+\alpha+\beta+1}{2} \cdot \frac{\Gamma(\nu+\alpha+\beta+1)}{\Gamma(\nu+\beta+1)} \cdot P_{\nu}^{(\alpha,\beta)}(x) =$$

$$= \frac{\Gamma(n+\alpha+\beta+2)}{(2n+\alpha+\beta+2)\Gamma(n+\beta+1)} \cdot \tag{3.1}$$

$$\cdot \frac{(n+\alpha+1)P_n^{(\alpha,\beta)}(x) - (n+1)P_{n+1}^{(\alpha,\beta)}(x)}{1-x}$$

But,

$$\left| \sum_{\nu=0}^{n} \frac{2\nu+\alpha+\beta+1}{2} \cdot \frac{\Gamma(\nu+\alpha+\beta+1)}{\Gamma(\nu+\beta+1)} \cdot P_\nu^{(\alpha,\beta)}(x) \right|$$

$$> \left| \sum_{\nu=0}^{n} \frac{2\nu+\alpha+\beta+1}{2} \cdot P_\nu^{(\alpha,\beta)}(x) \right|.$$

In view of (3.1), we have

$$\left| \sum_{\nu=0}^{n} (2\nu+\alpha+\beta+1) \cdot P_\nu^{(\alpha,\beta)}(x) \right|$$

$$< \left\{ \frac{(2n+\alpha+\beta+2)\Gamma(n+\beta+1)}{2\cdot\Gamma(n+\alpha+\beta+2)} \right\}^{-1} \cdot \frac{(n+\alpha+1)P_n^{(\alpha,\beta)}(x)-(n+1)P_{n+1}^{(\alpha,\beta)}(x)}{1-x}$$

$$< \frac{\Gamma(n+\alpha+\beta+2)}{\left(n+\frac{\alpha+\beta}{2}+1\right)\Gamma(n+\beta+1)} \cdot \frac{(n+\alpha+1)\left|P_n^{(\alpha,\beta)}(x)\right|+(n+1)\left|P_{n+1}^{(\alpha,\beta)}(x)\right|}{1-x}$$

$$< \frac{\Gamma(n+\alpha+\beta+2)}{\left(n+\frac{\alpha+\beta}{2}+1\right)\Gamma(n+\beta+1)} \cdot A(\varepsilon) \cdot \frac{\left\{ \frac{n+\alpha+1}{\sqrt{n}} + \frac{n+1}{\sqrt{n+1}} \right\}}{1-x}$$

$$< A(\varepsilon) n^{\frac{1}{2}+\alpha}. \tag{3.2}$$

Consider now the series,

$$\sum_{\nu=0}^{n} (2\nu+\alpha+\beta+1)^\mu P_\nu^{(\alpha,\beta)}(x),$$

where

$$0 < \mu < \frac{1}{2} - \alpha, \ 0 < \alpha < \frac{1}{2}.$$

Let

$$\sum_{\nu=0}^{n} (2\nu+\alpha+\beta+1)^\mu P_\nu^{(\alpha,\beta)}(x) = \sum_{\nu=0}^{n} C_\nu \varphi_\nu(x),$$

where $C_\nu = (2\nu + \alpha + \beta + 1)^{\mu-1}$; and

$$\varphi_\nu(x) = (2\nu + \alpha + \beta + 1)P_\nu^{(\alpha,\beta)}(x).$$

Put

$$\Phi_n(x) = \sum_{\nu=0}^{n} \varphi_\nu(x)$$

$$= \sum_{\nu=0}^{n} (2\nu + \alpha + \beta + 1)P_\nu^{(\alpha,\beta)}(x).$$

Using ABEL's transformation (5, p. 3), we have

$$\left| \sum_{\nu=0}^{n} C_\nu \varphi_\nu(x) \right|.$$

$$= \left| \sum_{\nu=0}^{n-1} \Phi_\nu(x) \{ (2\nu + \alpha + \beta + 1)^{\mu-1} - (2\nu + \alpha + \beta + 3)^{\mu-1} \} \right.$$

$$\left. + \Phi_n(x) \cdot (2n + \alpha + \beta + 1)^{\mu-1} \right|.$$

$$\leq \sum_{\nu=0}^{n-1} | \Phi_\nu(x) | \, | (2\nu + \alpha + \beta + 1)^{\mu-1} - (2\nu + \alpha + \beta + 3)^{\mu-1} |$$

$$+ | \Phi_n(x) | \, | (2n + \alpha + \beta + 1)^{\mu-1} |.$$

$$< A(\epsilon) \left\{ \sum_{\nu=0}^{u-1} \nu^{\frac{1}{2}+\alpha} | (2\nu + \alpha + \beta + 1)^{\mu-1} - (2\nu + \alpha + \beta + 3)^{\mu-1} | \right.$$

$$\left. + n^{\frac{1}{2}+\alpha} | (2n + \alpha + \beta + 1)^{\mu-1} | \right\}.$$

$$< A(\epsilon) \left\{ \sum_{n=0}^{\infty} n^{\frac{1}{2}+\alpha} (2n + \alpha + \beta + 1)^{\mu-2} \cdot B + 0(1) \right\},$$

whese B is a constant.

(3.3) $$< A(\varepsilon).$$

The series (1.1) is,

$$\sum_{n=0}^{\infty} a_n P_n^{(\alpha,\beta)}(x)$$

$$= \sum_{n=0}^{\infty} \frac{a_n}{(2n+\alpha+\beta+1)^{\mu}} \cdot (2n+\alpha+\beta+1)^{\mu} \cdot P_n^{(\alpha,\beta)}(x)$$

$$= \sum_{n=0}^{\infty} \alpha_n \{ C_n \varphi_n(x) \}, \text{ say.}$$

The sequence $\{\alpha_n\} \to 0$ and is of bounded total variation, and in view of (3.3), the series

$$\sum_{n=0}^{\infty} C_n \varphi_n(x)$$

has its partial sums uniformly bounded.

Hence the theorem is proved by virtue of ABEL's criterion for the uniform convergence of a series.

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