Bollettino
Unione Matematica Italiana

Leonard Carlitz

An integral for the product of two Laguerre polynomials.

Bollettino dell'Unione Matematica Italiana, Serie 3, Vol. **17** (1962), n.1, p. 25–28.

Zanichelli

<http://www.bdim.eu/item?id=BUMI_1962_3_17_1_25_0>

Bollettino dell'Unione Matematica Italiana, Zanichelli, 1962.

An integral for the product of two Laguerre polynomials

by LEONARD CARLITZ (a Durham, U. S. A.) (*)

Summary. - *The product* $L_m^{(\alpha)}(x)L_n^{(\beta)}(y)$ *is represented by a double integral.*

Let

$$L_m^{(\alpha)}(x) = \sum_{r=0}^{\infty} (-1)^r \frac{\Gamma(\alpha+m+1)}{(m-r\,!\,\Gamma(\alpha+r+1)} \frac{x^r}{r!} \tag{1}$$

denote the LAGUERRE polynomial of degree m. Then

$$L_m^{(\alpha)}(x)L_n^{(\beta)}(y) = \sum_{r=0}^{\infty}\sum_{s=0}^{n} (-1)^{r+s} \frac{x^r y^s}{r!\,s!} \frac{\Gamma(\alpha+m+1)\Gamma(\beta+n+1)}{(m+n-r-s)\Gamma(\alpha+\beta+r+s+1)} \cdot$$

$$\cdot \frac{(m+n-r-s)!}{(m-r)!\,(n-s)!} \frac{\Gamma(\alpha+\beta+rs+1)}{\Gamma(\alpha+r+1)\Gamma(\beta+s+1)}.$$

Since [4. p. 263]

$$\frac{\Gamma(\mu+\nu+1)}{\Gamma(\alpha+1)\Gamma(\beta+1)} = \frac{2^{\mu+\nu}}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{(\mu-\nu)\theta i} \cos^{\mu+\nu}\theta\, d\theta \qquad (\mu+\nu > -1),$$

it follows that

$$L_m^{(\alpha)}(x)L_n^{(\beta)}(y) = \frac{2^{\alpha+\beta+m+n}}{\pi^2} \Gamma(\alpha+m+1)\Gamma(\beta+n+1) \cdot$$

$$\cdot \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{\frac{\pi}{2}}^{\frac{\pi}{2}} e^{(m-u)\varphi i+(\alpha-\beta)\theta i} \cos^{\alpha+\beta}\theta \,\cdot$$

$$\cdot \sum_{k=0}^{m+n} (-1)^k \frac{\cos^{m+n-k}\varphi \cos^k\theta}{k!\,(m+n-k)!\,\Gamma(\alpha+\beta+k+1)}$$

$$\cdot \sum_{r+s=k} \binom{k}{r} x^r y^s e^{(r-s)(\theta-\varphi)i}.$$

(*) Pervenuta alla Segreteria dell U. M. I. il 19 dicembre 1961.

Using (1) this reduces to

$$L_m^{(\alpha)}(x)L_n^{(\beta)}(y) = \frac{2^{\alpha+\beta+m+n}}{\pi^2} \frac{\Gamma(\alpha+m+1)\Gamma(\beta+n+1)}{\Gamma(\alpha+\beta+m+n+1)} \cdot$$

$$(3) \qquad \cdot \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{(m-n)\varphi i+(\alpha-\beta)\theta i} \cos^{m+n}\varphi \cos^{\alpha+\beta}\theta \cdot$$

$$\cdot L_{m+n}^{(\alpha+\beta)}\left(\frac{(xe^{(\theta-\varphi)i} + ye^{-(\theta-\varphi)i}) \cos\theta}{\cos\varphi}\right) d\varphi d\theta.$$

In particular, when $x = y$ (3) reduces to

$$L_m^{(\alpha)}(x)L_n^{(\beta)}(x) = \frac{2^{\alpha+\beta+m+n}}{\pi^2} \frac{\Gamma(\alpha+m+1)\Gamma(\beta+n+1)}{\Gamma(\alpha+\beta+m+n+1)} \cdot$$

$$(4) \qquad \cdot \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(m-n)\varphi \cos(\alpha-\beta)\theta \cos^{m+n}\varphi \cos^{\alpha+\beta}\theta \cdot$$

$$\cdot L_{m+n}^{(\alpha+\beta)}\left(\frac{2x\cos(\theta-\varphi)\cos\theta}{\cos\varphi}\right) d\varphi d\theta.$$

This formula may be compared with WATSON's formula [3]:

$$\frac{L_m^{(\alpha)}(x)}{\Gamma(\alpha+m+1)} \frac{L_n^{(\alpha)}(x)}{\Gamma(\alpha+n+1} =$$

$$(5) \qquad = \frac{2^{m+n1}}{\pi\Gamma\left(\alpha+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)} \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{L_{m+n}^{(\alpha)}(x(1+\sec\varphi\cos\theta))}{\Gamma(\alpha+m+n+1)} \cdot$$

$$\cdot \sin^{2\alpha}\theta \cos^{m+n}\varphi \cos(m-n)\varphi d\theta d\varphi.$$

As pointed out by WATSON, (5) includes BAILEY's formula [2]

$$(6) \quad H_m(x)H_n(x) = \frac{2^{\frac{1}{2}(m+n)} m!\, n!}{(m+n)!} \int_0^{\pi} H_{m+n}(x\sqrt{1+\sec\varphi}) \cos^{\frac{1}{2}(m+n)}\varphi$$

$$\cos\frac{1}{2}(m-n)\varphi d\varphi,$$

where

$$(7) \qquad H_m(x) = \sum_{2r=m} (-1)^r \frac{m!}{r!\,(m-2r)!} (2x)^{m-2r}.$$

It is not clear how to obtain a result like (6) from (3) or (4). However using (7) and (2) we get

$$H_m(x) H_n(y) = \frac{m!\,n!}{\pi} \sum_{r=0}^{m} \sum_{s=0}^{n} (-1)^{r+s} \frac{(2x)^{m-2r}(2y)^{n-2s}}{r!\,s!\,(m+n-r-s)!} \cdot$$

$$\cdot \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{(m-n-2r+2s)\theta i} \cos^{m+n-2r+2s}\theta d\theta.$$

Since the formula

$$\frac{(m+n-2r-2s)!}{(m-2r)!\,(n-2s)!} = \frac{2^{m+n+2r-2s}}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{(m-n-2r+2s)\theta i} \cos^{m+n-2r-2s}\theta d\theta$$

is valid for all r, s provided that $2r + 2s \leq m + n$, we may extend the range of r, s and get

$$H_m(x) H_n(y) = \frac{m!\,n!}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{(m-n)\theta i} \sum_{k=0}^{m+n} (-1)^k \frac{(2\cos\theta)^{m+n-2k} x^{m-2k} y^{n-2k}}{k!\,(m+n-2k)!} \cdot$$

$$\cdot (x^2 e^{2\theta i} + y^2 e^{-2\theta i})^k d\theta.$$

Therefore

$$(8) \qquad H_m(x) H_n(y) = \frac{1}{\pi} \frac{m!\,n!}{(m+n)!} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{(m-n)\theta i} \frac{w^{m+n}}{x^n y^m} H_{m+n}\left(\frac{xy\cos\theta}{w}\right) d\theta,$$

where

$$w = (x^2 e^{2\theta i} + y^2 e^{-2\theta i})^{\frac{1}{2}}.$$

In particular, when $x=y$, (8) reduces to

$$H_m(x)H_n(y)$$

$$=\frac{2^{\frac{1}{2}(m+n)}}{\pi}\frac{m!\,n!}{(m+n)!}\int_0^{\pi}\cos^{\frac{1}{2}(m+n)}\theta\cos\frac{1}{2}(m-n)\theta H_{m+n}\left(\frac{x\cos\frac{1}{2}\theta}{\sqrt{2\cos\theta}}\right)d\theta,$$

which is a disguised version of (6).

In connection with (8) compare [1, formula (5.5)].

We remark that for the confluent hypergeometric function

$$\Phi(a;\ c;\ x)=\sum_{r=0}^{\infty}\frac{r!\,(c)_r}{(a)^r}x^r$$

we can prove that

$$\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}\frac{\Gamma(c+d+1)}{\Gamma(c+1)\Gamma(d+1)}\Phi(a;\ c+1;\ x)\Phi(b;\ d+1;\ y)=$$

$$=\frac{2^{c+d}}{\pi}\int_0^1\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}t^{a-1}(1-t)^{b-1}\cos^{c+d}\theta\ \cdot$$

$$\cdot\ \Phi(xte^{\theta i}+y(1-t)e^{-\theta i})d\theta dt.$$

REFERENCES

[1] W. A. AL-SALAN and L. CARLITZ, *Some finite summation formulas for the classical orthogonal polynomials*, « Rendiconti di Matematica e delle sue applicazioni », series 5, vol. 16 (1957), pp. 74-95.

[2] W. N. BAILEY, *An integral representation for the product of two Hermite polyomials*, Journal of the London Mathematical Society », vol. 13 (1938), pp. 202-203.

[3] G. N. WATSON, *A note on the polnomials of Heemite and Laguerre*, « Journal of the London Mathematical Society », vol. 13 (1938), pp. 204-209.

[4] E. T. WHITTAKER and G. N. WATSON, *A course of modern analysis*, 4 th edition, Cambridge, 1927.