Bollettino
Unione Matematica Italiana

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Bollettino dell'Unione Matematica Italiana, Serie 3, Vol. **17** (1962), n.1, p. 22–24.

Zanichelli

<http://www.bdim.eu/item?id=BUMI_1962_3_17_1_22_0>

On series of squares of Bessel functions.

Nota di S. K. CHATTERJEA (a Calcutta) (1) (*)

Summary. - *Some series of squares of Bessel functions are considered.*

The BESSEL function $J_n(x)$ of order n is defined for $n > -1$ and $-\infty < x < +\infty$ by the power series

(1) $$J_n(x) = \sum_{\nu=0}^{\infty} (-)^{\nu} \frac{(x/2)^{n+2\nu}}{\Gamma(n+\nu+1)\nu!}.$$

It satisfies the recurrence formula

(2) $$J_{n+1}(x) = \frac{2n}{x} J_n(x) - J_{n-1}(x); \qquad (n > 0).$$

LOMMEL's [1] series of squares of BESSEL functions is

(3) $$\frac{x^2}{4} \{ J_n^2(x) - J_{n-1}(x)J_{n+1}(x) \} = \sum_{k=0}^{\infty} (n+1+2k) J_{n+1+2k}^2(x); \qquad (n > 0)$$

which shows that the expression $J_n^2(x) - J_{n-1}(x)J_{n+1}(x)$ remains positive when $n > 0$ and $-\infty < x < +\infty$. Recently SZÀSZ [2] has established the inequality:

(4) $$J_n^2(x) - J_{n-1}(x)J_{n+1}(x) > \frac{1}{n+1} J_n^2(x);$$

$$(n > 0, -\infty < x < +\infty)$$

THIRUVENKATACHAR and NANJUNDIAH [3] pointed out that the inequality (4) becomes an equality only for $x = 0$ $(n > 0)$. We have proved in a recent note [4] that

(5) $$\left.\begin{aligned} \Delta_{n,1,2;x}(J) &< 0, \ -\infty < x < 0 \\ &= 0, \ x = 0 \\ &> 0, \ 0 < x < +\infty \end{aligned}\right\}$$

where $\Delta_{n,1,2;x}(J) \equiv J_{n+1}(x)J_{n+2}(x) - J_n(x)J_{n+3}(x)$.

(*) Pervenuta alla Segreteria dell'U. M. I. il 18 dicembre 1961.

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Moreover we have deduced in that note the following series of squares of BESSEL functions for $\Delta_{n-2,1,2;x}(J)$:

$$\text{(6)}\qquad \frac{x}{4n(n-1)}\Delta_{n-2,1,2;x}(J) = \sum_{k=0}^{\infty} \frac{J_{n+k}^2(x)}{(n+k-1)(n+k+1)}$$

Now we first obtain the following series of squares of BESSEL functions for $x\Delta_{n,1,2;x}(J) + 2J_{n+1}^2(x)$:

$$\text{(7)}\qquad \frac{x^2}{2^3(n+2)}\left[x\Delta_{n,1,2;x}(J) + 2J_{n+1}^2(x)\right] = \sum_{k=0}^{\infty} (n+2+2k)J_{n+2+2k}^2(x).$$

For we immediately derive from the recursion relation (2):

$$\text{(8)}\qquad x\Delta_{n,1,2;x}(J) = 2\left[(n+2)\left\{J_{n+1}^2(x) - J_n(x)J_{n+2}(x)\right\} - J_{n+1}^2(x)\right]$$

Thus it follows from (8) and (3) that

$$x\Delta_{n,1,2;x}(J) + 2J_{n+1}^2(x) = \frac{2^3(n+2)}{x^2}\sum_{k=0}^{\infty} (n+2+2k)J_{n+2+2k}^2(x).$$

Next changing n into $n-2$ in (7) we get

$$\text{(9)}\qquad \frac{x^2}{2^3 n}\left[x\left\{J_{n-1}(x)J_n(x) - J_{n-2}(x)J_{n+1}(x)\right\} + 2J_{n-1}^2(x)\right] = \sum_{k=0}^{\infty} (n+2k)J_{n+2k}^2(x).$$

Now putting $n=\frac{1}{2}$ and using the well-known formulae

$$\text{(10)}\quad \begin{cases} J_{\frac{1}{2}}(x) = \left(\dfrac{2}{\pi x}\right)^{\frac{1}{2}} \operatorname{Sin} x; \quad J_{-\frac{1}{2}}(x) = \left(\dfrac{2}{\pi x}\right)^{\frac{1}{2}} \cos x \\[2ex] J_{\frac{3}{2}}(x) = \left(\dfrac{2}{\pi x}\right)^{\frac{1}{2}}\left(\dfrac{\operatorname{Sim} x}{x} - \cos x\right); \quad J_{-\frac{3}{2}}(x) = \left(\dfrac{2}{\pi x}\right)^{\frac{1}{2}}\left(\dfrac{-\cos x}{x} - \operatorname{Sin} x\right) \end{cases}$$

we derive from (9) that

$$(11) \qquad \frac{1}{2\pi}\left(x + \frac{\mathrm{Sin}\ 2x}{2}\right) = \sum_{k=0}^{\infty}\left(2k + \frac{1}{2}\right) J^2_{2k+\frac{1}{2}}(x)\,;$$

which may be compared with the formula [1, p 152].

If in addition we put $n = 2$ in (9) we obtain the result

$$(11) \qquad \frac{x^2}{16}[x \{ J_1(x)J_2(x) - J_0(x)J_3(x) \} + 2J_1^2(x)] = \sum_{k=0}^{\infty} (2k + 2)J^2_{2k+2}(x).$$

Now from (5) and (7) we may remark that

$$(13) \qquad \sum_{k=0}^{\infty} (n + 2 + 2k)J^2_{n+2+2k}(x) \geq \frac{x^2}{4(n+2)} J^2_{n+1}(x),$$

from which we at once derive

$$(14) \qquad \sum_{k=0}^{\infty} (2k + 1)J^2_{2k+1}(x) \geq \frac{x^2}{4} J_0^2(x),$$

which may well be compared with the result [1, p 152 foımula (4)]:

$$\frac{x^2}{4}[J_0^2(x) + J_1^2(x)] = \sum_{k=0}^{\infty} (2k + 1)J^2_{2k+1}(x).$$

Lastly by putting $n = \frac{1}{2}$ in (6) and using (10) we derive

$$(15) \qquad \frac{1}{4\pi x^2}[2x \cos 2x - \mathrm{Sin}\ 2x] = \sum_{k=0}^{\infty} \frac{J^2_{k+\frac{1}{2}}(x)}{(2k-1)(2k+3)}.$$

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