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Remarks on triangular and tetrahedral numbers.

Bollettino dell'Unione Matematica Italiana, Serie 3, Vol. **17** (1962), n.1, p. 20–21.

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<http://www.bdim.eu/item?id=BUMI_1962_3_17_1_20_0>

Remarks on triangular and tetrahedral numbers

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M. SATYANARAYANA proved in [4] that no triangular number > 1 is of the form $2^{2k-1} - 1$ and that 3 is the only triangular number of the form $2^k + 1$. G. BROWKIN and A. SCHIZEL proved in [1] that all triangular numbers of the form $2^k - 1$ are $2^1 - 1$, $2^2 - 1$, $2^4 - 1$ and $2^{12} - 1$. The equivalent result was established earlier by T. NAGELL [2] see also [3].

We prove here three following theorems.

I. If p is an odd prime number, then the only triangular numbers of the form $p^k + 1$ are $5^1 + 1$, $3^2 + 1$ and $3^3 + 1$.

II. The only tetrahedral number of the form $2^k - 1$ is 1.

III. There is no tetrahedral number of the form $2^k + 1$.

PROOF OF 1. - Let

$$\frac{n}{2}(n + 1) = p^k + 1.$$

Then

$$n^2 + n - 2 = 2 \cdot p^k, \quad (n - 1)(n + 2) = 2 \cdot p^k, \quad n + 2 \geq 3$$

and $n - 1$, $n + 2$ are neither both even, nor both odd. Thus, if

$$p \neq 3, \quad (n - 1,\ n + 2) = (n - 1,\ 3) = 1$$

and $n - 1 = 2$, $n = 3$. Hence we get the triangular number $6 = 5^1 + 1$. If $p = 3$, then

$$n - 1 = 3^a, \quad n + 2 = 2 \cdot 3^b \quad \text{or} \quad n - 1 = 2 \cdot 3^a, \quad n + 2 = 3^b.$$

In the first case

$$3 = 2 \cdot 3^b - 3^a, \quad 1 = 2 \cdot 3^{b-1} - 3^{a-1}, \quad a = b = 1, \quad k = 2.$$

(*) Pervenuta alla Segreteria dell' U. M. I. il 25 marzo 1962.

In the second case

$$3 = 3^b - 2\cdot 3^a, \quad 1 = 3^{b-1} - 2\cdot 3^{a-1}, \quad a = 1, \quad b = 2, \quad k = 3.$$

PROOF OF II. - Let

$$\frac{n}{6}(n+1)(n+2) = 2^k - 1.$$

Then

$$n^3 + 3n^2 + 2n + 6 = 3\cdot 2^{k+1}, \quad (n+3)(n^2+2) = 3\cdot 2^{k+1}.$$

It is evident that $n+3 \geq 4$. $n^2+2 \geq 3$ and that one of the factors of the left-hand side is even and the other is odd. Because $3\cdot 2^{k+1}$ can be iniquely repiesented as the product of odd and even positive integers (both ≥ 3), then $n^2 + 2 = 3$.

Hence $n = 1$, g.e.d.

PROOF OF III. - Let

$$\frac{n}{6}(n+1)(n+2) = 2^k + 1.$$

Then

$$n^3 + 3n^2 + 2n - 6 = 3\cdot 2^{k+1}. \quad (n-1)(n^2+4n+6) = 3\cdot 2^{k+1}.$$

In the same way as in the proof of II we get $n - 1 = 1$ or 3, which is impossible.

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