BOLLETTINO
UNIONE MATEMATICA ITALIANA

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On a formula of Al Salam.

Bollettino dell'Unione Matematica Italiana, Serie 3, Vol. **16** (1961), n.4, p. 425–427.
Zanichelli
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Bollettino dell'Unione Matematica Italiana, Zanichelli, 1961.

On a formula of Al Salam

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Summary. - *Some integral formulae of* W. A. AL-SALAM *are corrected.*

In W. A. AL-SALAM'S note « Some integral formulas for certain TURAN expressions », appearing in the « Bulletin of the College of Science » [1], he has proved the following integral transform:

$$(1) \qquad \int_{-\infty}^{\infty} e^{-x^2}\Delta_n(H)\cos(2xy)dx = 2^{n+1}n!\,\sqrt{\pi}L_n^{(1)}(2y^2),$$

where

$$\Delta_n(H) \equiv H_n^2(x) - H_{n+1}(x)H_{n-1}(x).$$

But this formula is misprinted and we deduce the correct formula as follows: First we notice that [2]

$$(2) \qquad \int_0^{\infty} e^{-\frac{1}{2}x^2}[He_n(x)]^2\cos(xy)dx = (\pi/2)^{\frac{1}{2}}n!\,e^{-\frac{1}{2}y^2}L_n(y^2)$$

$$(3) \qquad \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2}(x-y)^2\right][He_n(x)]^2dx = (2\pi)^{\frac{1}{2}}n!\,L_n(-y^2)$$

Now since $He_n(x) = 2^{-\frac{1}{2}n}H_n(2^{-\frac{1}{2}}x)$, we can rewrite the formulae (2) and (3) in the following fashion:

$$(4) \qquad \int_{-\infty}^{\infty} e^{-x^2}H_n^2(x)\cos(2xy)dx = \pi^{\frac{1}{2}}2^n n!\,e^{-y^2}L_n(2y^2)$$

$$(5) \qquad \int_{-\infty}^{\infty} \exp[-(x-y)^2]H_n^2(x)dx = \pi^{\frac{1}{2}}2^n n!\,L_n(-2y^2)$$

(*) Pervenuta alla Segreteria dell'U. M. I. il 16 ottobre 1961.

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Next if $\Delta_n(H) = H_{n+1}^2(x) - H_n(x)H_{n+2}(x)$, we have [3]

$$\Delta_n(H) = 2^{n+1}n!\sum_{m=0}^{n}\frac{H_m^2(x)}{2^m m!}. \tag{6}$$

Thus it follows from (4) and (5) that

$$\int_{-\infty}^{\infty} e^{-x^2}[H_{n+1}^2 - H_nH_{n+2}]\cos(2xy)dx = 2^{n+1}n!\,\pi^{\frac{1}{2}}e^{-y^2}\sum_{m=0}^{n} L_m(2y^2) \tag{7}$$

$$\int_{-\infty}^{\infty} \exp[-(x-y)^2][H_{n+1}^2 - H_nH_{n+2}]dx = 2^{n+1}n!\,\pi^{\frac{1}{2}}\sum_{m=0}^{n} L_m(-2y^2) \tag{8}$$

Now we know that

$$\sum_{m=0}^{n} L_m^{(\alpha)}(x) = L_n^{(\alpha+1)}(x).$$

Therefore we derive ultimately

$$\int_{-\infty}^{\infty} e^{-x^2}[H_{n+1}^2 - H_nH_{n+2}]\cos(2xy)dx = 2^{n+1}n!\,\pi^{\frac{1}{2}}e^{-y^2}L_n^{(1)}(2y^2) \tag{9}$$

$$\int_{-\infty}^{\infty} \exp[-(x-y)^2][H_{n+1}^2 - H_nH_{n+2}]dx = 2^{n+1}n!\,\pi^{\frac{1}{2}}L_n^{(1)}(-2y^2). \tag{10}$$

Thus the formulae (9) and (10) can well be compared with (1).

In particular we notice, on using $n = 0$ in (9) and (10), the well-known formulae:

$$\int_{-\infty}^{\infty} e^{-x^2}\cos(2xy)dx = \sqrt{\pi}e^{-y^2} \tag{11}$$

$$\int_{-\infty}^{\infty} e^{-(x-y)^2}dx = \sqrt{\pi}. \tag{12}$$

Again by employing $y = 0$ in (9) and (10), we derive the result [4]

$$(13) \qquad \int_{-\infty}^{\infty} e^{-x^2} [H_{n+1}^2 - H_n H_{n+2}] dx = 2^{n+1}(n+1)! \sqrt{\pi}.$$

Lastly AL-SALAM [1, p 19] has also given the following inversion formula for (1):

$$(14) \qquad \Delta_n(H) = \frac{2^{n+1} n!}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp[-(y + ix)^2] L_n^{(1)}(2y^2) dy,$$

where

$$\Delta_n(H) = H_n^2(x) - H_{n+1}(x) H_{n-1}(x).$$

We would like to point out that (14) does not follow from (1). However, the inversion formula (14) is true, the only necessary change to be made is that

$$\Delta_n(H) = H_{n+1}^2(x) - H_n(x) H_{n+2}(x).$$

Indeed such an inversion formula (14) follows from the FOURIER cosine transform (9).

REFERENCES

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