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The relationship of the Hermite to the Laguerre Polynomials.

by LEONARD CARLITZ (a Durham, U. S. A.) (*)

Summary. - *Given two sets of orthogonal polynomials* $\{f_n(x)\}$, $\{g_n(x)\}$ *and put*

$$\Phi_{2m}(x) = f_m(x^2), \quad \Phi_{2m+1}(x) = xg_m(x^2).$$

We seek conditions such that the set of polynomials $\{\Phi_n(x)\}$ *will be orthogonal.*

It is well known (see for example [2, p. 102] that the Hermite polynomials can be expressed in terms of the Laguerre polynomials with the parameters $\pm\frac{1}{2}$ by means of the formulas

$$H_{2m}(x) = (-1)^m 2^{2m} m!\ L_m^{\left(-\frac{1}{2}\right)}(x^2),$$

$$H_{2m+1}(x) = (-1)^m 2^{2m+1} m!\ xL_m^{\left(\frac{1}{2}\right)}(x^2).$$

This suggests the following problem.

Given two sets of orthogonal polynomials $\{f_n(x)\}$. $\{g_n(x)\}$, put

$$\Phi_{2m}(x) = f_m(x^2), \quad \Phi_{2m+1}(x) = xg_m(x^2). \tag{1}$$

We may ask under what conditions will the set of polynomials $\{\Phi_n(x)\}$ be orthogonal.

It will be convenient to assume, as we may without loss of generality, that the highest coefficients of $f_n(x)$ and $g_n(x)$ are equal to 1. We recall [2, p. 41] that the orthogonality of the set $\{f_n(x)\}$ implies the existence of a set of constants A_n, B_n such that

$$f_{n+1}(x) = (x + A_n)f_n(x) - B_nf_{n-1}(x) \qquad (n=1, 2, 3, ...), \tag{2}$$

where $B_n > 0$. Moreover the existence of such a recurrence together with $B_n > 0$ implies the orthogonality of $\{F_n(x)\}$ (Favard [1]). This suggests a slight generalization of the above problem. Let the $f_n(x)$ satisfy (2), where nothing is assumed about the sign

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of B_n; also let the $g_n(x)$ satisfy

(3) $$g_{n+1}(x) = (x + A'_n)g_n(x) - B'_n g_{n-1}(x).$$

Define the polynomials $\Phi_n(x)$ by means of (1). Since $\Phi_n(-x) = (-1)^n\Phi(x)$, we require that

(4) $$\Phi_{n+1}(x) = x\Phi_n(x) - C_n\Phi_{n-1}(x) \qquad (n=1,\ 2,\ 3,\ \ldots).$$

For $n = 2m$, (4) and (1) imply

$$xg_m(x^2) = xf_m(x^2) - C_{2m}xg_{m-1}(x^2),$$

so that

(5) $$g_m(x) = f_m(x) - C_{2m}g_{m-1}(x)..$$

Similarly, by taking $n = 2m + 1$, we get

(6) $$f_{m+1}(x) = xg_m(x) - C_{2m+1}f_m(x).$$

By (6) we have

$$xg_m(x) = f_{m+1}(x) - C_{2m+1}f_m(x).$$

Substituting in (5) we find that

(7) $$f_{m+1}(x) = (x - C_{2m} - C_{2m+1})f_m(x) - C_{2m}C_{2m-1}f_{m-1}(x).$$

Comparison of (7) with (2) yields

(8) $$A_m = -C_{2m} - C_{2m+1}, \quad B_m = C_{2m}C_{2m-1}.$$

If we eliminate $f_m(x)$ (5) and (6) we get

$$g_{m+1}(x) = (x - C_{2m+1} - C_{2m+2})g_m(x) - C_{2m}C_{2m+1}g_{m-1}(x),$$

so that

(9) $$A'_m = -C_{2m+1} - C_{2m+2}, \quad B'_m = C_{2m}C_{2m+1}.$$

This evidently proves

THEOREM 1. - *Let* $\{\Phi_n(x)\}$ *denote a set of polynomials, such that* $\Phi_0(x) = 1$, $\Phi_1(x) = x$ *and*

$$\Phi_{n+1}(x) = x\Phi_n(x) - C_n\Phi_{n-1}(x) \qquad (n=1,\ 2,\ 3,\ \ldots).$$

Then (8) *and* (9) *uniquely determine two sets of polynomials* $\{f_n(x)\}$, $\{g_n(x)\}$ *that satisfy* (1), (2) *and* (3).

If we make the stronger assumption that the $\Phi_n(x)$ form an orthogonal set, with weight function $w(x)$, this result can be proved very rapidly as follows. We have

$$(10) \qquad \int_{-a}^{a} \Phi_m(x)\Phi_n(x)w(x)dx = k_m\delta_{m,n}.$$

Since $\Phi_n(-x) = (-1)^n\Phi_n(x)$, we may use (1) to define the polynomials $f_n(x)$, $g_n(x)$. Also (10) may be replaced by

$$(11) \qquad 2\int_{0}^{a} \Phi_m(x)\Phi_n(x)w(x)dx = k_n\delta_{m,n}.$$

Hence by (1)

$$2\int_{0}^{a} f_m(x^2)f_n(x^2)w(x)dx = k_{2n}\delta_{m,n},$$

so that

$$\int_{0}^{a} f_m(x)f_n(x)x^{-\frac{1}{2}}w(x^{\frac{1}{2}})dx = k_{2n}\delta_{m,n}.$$

Similarly by raplacing m and n in (11) by $2m+1$ and $2n+1$, respectively, we get

$$\int_{0}^{a} g_m(x)g_n(x)x^{\frac{1}{2}}w(x^{\frac{1}{2}})dx = k_{2n+1}\delta_{m,n}.$$

This proves

THEOREM 2. - *Let* $\{\Phi_n(x)\}$ *denote a set of polynomials orthogonal over the interval* $(-a, a)$ *with weight function* $w(x)$ *and such that*

$$\Phi_n(-x) = (-1)^n\Phi(n).$$

Then the sets of polynomials $\{f_n(x)\}$, $\{g_n(x)\}$ *uniquely determined by* (1) *are orthogonal over the interval* $(0, a)$ *with weight functions* $x^{-\frac{1}{2}}w(x^{\frac{1}{2}})$ *and* $xw(x^{\frac{1}{2}})$, *respectively.*

For example for the Legendre polynomial $P_n(x)$ we may put

$$P_{2m}(x) = R_m(x^2), \quad P_{2m+1}(x) = xS_m(x^2).$$

Then the $R_n(x)$ are orthogonal over (0, 1) with weight function $x^{-\frac{1}{2}}$, while the $S_n(x)$ are orthogonal over (0, 1) with weight function $x^{\frac{1}{2}}$.

Conversely we may assume that either the $f_n(x)$ or the $g_n(x)$ are assigned sets of orthogonal polynomials over the interval $(0, a)$.

Assume that

$$\int_0^a f_m(x)f_n(x)w_1(x)dx = k'_n\delta_{m,n}. \tag{12}$$

Then if there exist sets of polynomials $\{g_n(x)\}$, $\{\Phi_n(x)\}$ such that (1) and (10) are satisfied and

$$\int_0^a g_m(x)g_n(x)w_2(x)dx = k''_n\delta_{m,n}, \tag{13}$$

it follows from the last theorem that

$$w_2(x) = xw_1(x), \quad w(x) = xw_1(x^2). \tag{14}$$

We may now state

THEOREM. 3. – *Let* $\{f_n(x)\}$, $\{g_n(x)\}$, $\{\Phi_n(x)\}$ *be three sets of polynomials that satisfy* (10), (12) *and* (13), *respectively. Then the polynomials satisfy* (1) *if and only if the weight functions satisfy* (14). *Moreover if any one of the sets is given the other two sets are uniquely determined.*

We may return to the point of view of Theorem 1. The situation is now rather different from that described in Theorem 3. For example let

$$f_n(x) = x^n \qquad (n = 0, 1, 2, \ldots),$$

so that

$$\Phi_{2n}(x) = x^{2n} \qquad (n = 0, 1, 2, \ldots).$$

Then by (4)

$$\Phi_{2n+2}(x) = x\Phi_{2n+1}(x) - C_{2n+1}\Phi_{2n}(x),$$

so that

$$(15) \qquad \Phi_{2n+1}(x) = x^{2n+1} + C_{2n+1}x^{2n-1}.$$

Again using (4) we get

$$\Phi_{2n+1}(x) = x\Phi_{2n}(x) - C_{2n}\Phi_{2n-1}(x),$$

so that by (15)

$$x^{2n+1} + C_{2n+1}x^{2n-1} = x^{2n+1} - C_{2n}(x^{2n-1} + C_{2n-2}x^{2n-3}).$$

This requires

$$(16) \qquad C_{2n+1} = - C_{2n}, \quad C_{2n}C_{2n-1} = 0.$$

Since

$$\Phi_2(x) = x^2 - C_1, \quad C_1 = 0,$$

we can satisfy (16) by taking

$$C_{4r+1} = C_{4r} = 0, \quad C_{4r+3} = - C_{4r+2}$$

with C_{4r+2} arbitrary. Thus the polynomial set $\{\Phi_n(x)\}$ is not uniquely determined by $\{f_n(x)\}$.

In the next place if we take

$$f_1(x) = x, \quad f_2(x) = x^2 - 1, \quad f_n(x) = x^n \qquad (n=3, 4, 5, \ldots),$$

it is easily verified that (2) is satisfied. By (1) we have $\Phi_2(x)=x^2$, $\Phi_4(x) = x^4 - 1$. Then by (4) whith $n = 3$ we should have

$$x^4 - 1 = x\Phi_3(x) - C_3x^2.$$

It is obviously impossible to satisfy this relation.

From these two examples we see that for a given sequence $\{f_n(x)\}$ that satisfies (2) it may be impossible to find a sequence $\{\Phi_n(x)\}$ that satisfies both (4) and the first part of (1) or alternatively there may be infinitely many such aequences. The question of what conditions on $\{f_n(x)\}$ guarantee the existence of $\{\Phi_n(x)\}$ is left open.

REFERENCES

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