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Zanichelli

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Note on H_2 summability of Fourier series.

By O. P. VARSHNEY (a Sagar, India) (*).

Summary. - *In this paper, a criterion for H_2 summability of Fourier series, similar to that of Wang* (2), *has been established. The condition of Wang has been weakened, and an order condition on Fourier coefficients has been imposed.*

1. Let $S(f)$ i. e.,

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt) \tag{1.1}$$

be the FOURIER series of an integrable function $f(t)$, periodic with period 2π, and let $\Phi(t) = \frac{1}{2}\{f(x+t) + f(x-t) - 2s\}$.

In this note we prove the following theorem.

THEOREM. - *If* (*i*) $\int_0^t |\Phi(u)|\, du = 0\left(t \log^{-\frac{1}{2}}\left(\frac{1}{t}\right)\right)$ *as* $t \to 0$, *and* (*ii*) *the coefficients of* $S(f)$ *are* $O(n^{-\delta})$, $\delta > 0$, *then* $S(f)$ *is summable* H_2 *to sum* s *for* $t = x$.

Since instead of $S(f)$ we may consider $S(\Phi)$, let us assume that $x = 0$, $f(x) = f(-x)$, so that $\Phi(t) = f(t)$. For the sake of convenience we suppose that $a_0 = 0$. We write $r = \frac{\delta}{2}$ and $s_n = \sum_1^n a_\nu$.

In order to prove this theorem, we require the following lemma.

LEMMA. - *If* $|a_n| < n^{-\delta}$, $0 < \delta < \frac{1}{2}$, *and*

$$F(t) = \int_0^t |f(u)|\, du = 0(t) \text{ as } t \to 0 \tag{1.2}$$

then

$$\left(\frac{1}{(n+1)}\sum_{\nu=1}^{n} s_\nu^2\right)^{1/2} \leq \left(\frac{4}{\pi^2(n+1)}\int_{1/n}^{n^{-r}} \frac{f(t)}{t^2}\, dt \int_{1/n}^{t} f(u)\, \frac{\sin n(u-t)}{u-t}\, du\right)^{1/2} + 0(1).$$

(*) Pervenuta alla Segreteria dell'U. M. I. l'11 settembre 1961.

Proof. - By (1.2) we obtain

$$s_\nu = \frac{2}{\pi}\int_{n^{-1}}^{\pi} f(t)\,\frac{\sin \nu t}{t}\,dt + 0(1) \qquad (\nu \leq n)$$

$$= \frac{2}{\pi}\left(\int_{n^{-1}}^{n^{-r}} + \int_{n^{-r}}^{\pi}\right) f(t)\,\frac{\sin \nu t}{t}\,dt + 0(1)$$

$$(1.3) \qquad = P + Q + 0(1),$$

say. Using the arguments of ZYGMUND [1], we have

$$|\,Q\,| = \left|\sum_{k=1}^{\infty} a_k \,\frac{2}{\pi}\int_{n^{-r}}^{\pi} \frac{\sin \nu t \cos kt}{t}\,dt\right|$$

$$\leq O(\nu^{-\delta} \log n) + \frac{4}{\pi}\sum_{k=1}^{\nu-1} \frac{k^{-\delta} n^r}{\nu - k} + \frac{4}{\pi}\sum_{k=\nu+1}^{\infty} \frac{k^{-\delta} n^r}{k - \nu}$$

$$(1.4) \qquad = O(n^r \nu^{-\delta}) + \frac{4}{\pi}\,Q_1 + \frac{4}{\pi}\,Q_2\,,$$

say. Now

$$(1.5) \qquad Q_1 < \frac{n^r}{\nu/2}\sum_{k=1}^{\left[\frac{\nu}{2}\right]} k^{-\delta} + n^r\left(\frac{\nu}{2}\right)^{-\delta}\sum_{k=\left[\frac{\nu}{2}\right]+1}^{\nu-1} \frac{1}{\nu - k} = O\left(\frac{n^r \log n}{\nu\delta}\right)$$

and

$$(1.6) \qquad Q_2 < \frac{n^r}{\nu^\delta}\sum_{k=\nu+1}^{2\nu} \frac{1}{k - \nu} + n^r \sum_{k=2\nu+1}^{\infty} \frac{k^{-\delta}}{k/2} = O\left(\frac{n^r \log n}{\nu\delta}\right).$$

Combining (1.3), (1.4), (1.5) and (1.6) we obtain for $\nu \leq n$

$$s_\nu = \frac{2}{\pi}\int_{n^{-1}}^{n^{-r}} f(t)\,\frac{\sin \nu t}{t}\,dt + 0(1) + O\left(\frac{n^r \log n}{\nu\delta}\right)$$

$$= \sigma_\nu + 0(1) + O(n^r \log n\, \nu^{-\delta}), \text{ say.}$$

(1) ZYGMUND, *Trigonometrical series*, (1935) p. 35.

Hence we get

$$\left(\frac{1}{n+1}\sum_1^n s_\nu^2\right)^{1/2} \leq \left(\frac{1}{n+1}\sum_{\nu=1}^n \sigma_\nu^2\right)^{1/2} + 0(1) + O\left(\frac{n^{2r}\log^2 n}{n+1}\sum_1^n \nu^{-2\delta}\right)^{1/2}$$

$$= \left(\frac{1}{n+1}\sum_{\nu=1}^n \sigma_\nu^2\right)^{1/2} + 0(1),$$

The proof is now similar to WANG's lemma [2].

2. *Proof of the theorem.* It follows from the hypothesis that

$$\int_{n^{-1}}^{t} f(u)\frac{\sin n(u-t)}{u-t}\,du = O\left(n\int_0^t |f(u)|\,du\right) = 0\left(\frac{nt}{\log^{1/2}\left(\frac{1}{t}\right)}\right).$$

Hence we get

$$\int_{n^{-1}}^{n^{-r}} \frac{f(t)}{t^2}\,dt \int_{1/n}^{t} f(u)\frac{\sin n(u-t)}{u-t}\,du$$

$$= 0\left(n\int_{n^{-1}}^{n^{-r}} \frac{|f(t)|}{t\log^{1/2}\left(\frac{1}{t}\right)}\,dt\right)$$

$$= 0\left[\left\{\frac{nF(t)}{t\log^{1/2}\left(\frac{1}{t}\right)}\right\}_{n^{-1}}^{n^{-r}} + \int_{n^{-1}}^{n^{-r}} \frac{nF(t)}{t^2\log^{1/2}\left(\frac{1}{t}\right)}\,dt - \frac{1}{2}\int_{n^{-1}}^{n^{-r}} \frac{nF(t)}{t^2\log^{3/2}\left(\frac{1}{t}\right)}\,dt\right]$$

$$= 0(n) + 0(n)\left(\int_{n^{-1}}^{n^{-r}} \frac{1}{t\log\left(\frac{1}{t}\right)}\,dt\right) = 0(n).$$

The theorem now follows from the lemma.

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[2] WANG, «Duke Math. Journal», vol. 12 (1945) pp. 77-8.