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Zanichelli

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On the matrix summability of the derived Fourier series.

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Summary. - *In this paper, a sufficient condition for the matrix summability of the Fourier series of the symmetric derivative of a function* f(x)L *is proved. The summability chosen includes harmonic summability as a particular case.*

1. Let $f(\theta)$ be a periodic function with period 2π and integrable (L) over $(-\pi, \pi)$. Let the FOURIER series associated with $f(\theta)$ be

$$\frac{1}{2}a_0 + \sum_{1}^{\infty}(a_n \cos n\theta + b_n \sin n\theta). \tag{1.1}$$

Then the differentiated series of (1.1) at $\theta = x$ is

$$\sum_{1}^{\infty} n(b_n \cos nx - a_n \sin nx) \equiv \sum_{1}^{\infty} nB_n(x). \tag{1.2}$$

We write

$$\psi(t) = f(x + t) - f(x - t); \qquad g(t) = \frac{\psi(t)}{4\sin\frac{1}{2}t} - C, \tag{1.3}$$

where C is a function of x.

Let $\Lambda = \{d_{n,k}\}$, $n.\ k = 0, 1, 2, \dots$, and $d_{n,0} = 1$ be a triangular matrix. Let $\Delta d_{n,k} = d_{n,k} - d_{n,k+1}$. An infinite series Σu_n with partial sums $s(n)$ is said to be summable (Λ) to the value s if $\sigma(n) \to s$ as $n \to \infty$, where

$$\sigma(n) = \sum_{k=0}^{n} \Delta d_{n,k}s(k). \tag{1.4}$$

The conditions of the regularity of the method of summation are

$$\begin{aligned} &(i) \lim_{n\to\infty} d_{n,k} = 1 \text{ for every fixed } k, \text{ and} \\ &(ii) \Sigma \ |\Delta d_{n,k}| \leq K \text{ independently of } n. \end{aligned} \tag{1.5}$$

The object of this note is to prove the following theorem:

TEOREM. - *If* $g(t)$ *is of bounded variation in* $(0, \pi)$ *and* $g(t) \to 0$ *as* $t \to 0$, *then the series* $\Sigma nB_n(x)$ *is summable* (Λ) *to the*

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value C, provided that (Λ) *is regular and*

$$\lim_{n\to\infty} \sum_{k=0}^{n} |\Delta^2 d_{n,k}| = 0. \tag{1.6}$$

This theorem includes as a perticular case a recent theorem of RATH [1] on Harmonic summability of the series (1.2).

Our method of summability being regular, it is easy to see that there 's no loss of generality in assuming $C=0$ [1].

2. Proof of the theorem:

Using (1.2) we have

$$kB_k = \frac{1}{\pi}\int_0^\pi \psi(t) k \sin kt dt.$$

Hence by (1.3) we obtain

$$s(n) = \frac{1}{\pi}\int_0^\pi \psi(t)\left(\sum_{k=1}^{n} k\sin kt\right) dt$$

$$= -\frac{1}{2\pi}\int_0^\pi \psi(t) d\left(\frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\frac{1}{2}t}\right)$$

$$= \frac{1}{2\pi}\int_0^\pi \frac{\psi(t)}{2\sin\frac{t}{2}} \sin\left(n+\frac{1}{2}\right)t \cot\frac{t}{2} dt -$$

$$-\frac{1}{2\pi}\int_0^\pi \psi(t)\left(n+\frac{1}{2}\right)\cos\left(n+\frac{1}{2}\right)t/\sin\frac{t}{2} dt$$

$$= \frac{1}{\pi}\int_0^\pi g(t)\sin\left(n+\frac{1}{2}\right)t/\tan\frac{t}{2} dt -$$

$$-\frac{2}{\pi}\int_0^\pi g(t)\left(n+\frac{1}{2}\right)\cos\left(n+\frac{1}{2}\right)t dt$$

$$= s_1(n) + s_2(n), \text{ say.} \tag{2.1}$$

Hence, by (1.4) we have

$$\sigma(n) = \frac{1}{\pi} \sum_{k=0}^{n} \left[\int_0^\pi \Delta d_{n,k} \sin\left(k+\frac{1}{2}\right) t g(t) \cot\frac{t}{2}\, dt \right]$$

$$- \frac{2}{\pi} \sum_{k=0}^{n} \left[\int_0^\pi \Delta d_{n,k} \left(k+\frac{1}{2}\right) \cos\left(k+\frac{1}{2}\right) t g(t) dt \right]$$

(2.2) $$= \sigma_1(n) + \sigma_2(n), \text{ say.}$$

Since $g(t)$ is of bounded variation in $(0, \pi)$ and $g(t) \to 0$ as $t \to 0$, $s_1(n) = 0(1)$ as $n \to \infty$, the proof being similar to that of JORDAN's convergence criterion for FOURIER series [2]. By regularity of our method of summation it follows that

(2.3) $$\sigma_1(n) = 0(1) \text{ as } n \to \infty.$$

Now integrating by parts and making use of (1.6) we have

$$\sigma_2(n) = -\frac{2}{\pi} \int_0^\pi \left(\sum_{k=0}^{n} \Delta d_{n,k} \left(k+\frac{1}{2}\right) \cos\left(k+\frac{1}{2}\right) t \right) g(t)\, dt$$

$$= -\frac{2}{\pi} g(\pi) \left(\sum_{k=0}^{n} \Delta d_{n,k} \sin\left(k+\frac{1}{2}\right)\pi \right) +$$

$$+ \frac{2}{\pi} \int_0^\pi \left(\sum_{k=0}^{n} \Delta d_{n,k} \sin\left(k+\frac{1}{2}\right) t \right) dg(t)$$

$$= -\frac{2g(\pi)}{\pi} \left(\sum_{k=0}^{n} (-1)^k \Delta d_{n,k} \right) +$$

$$+ \frac{2}{\pi} \int_0^\pi \left(\sum_{k=0}^{n} \Delta d_{n,k} \sin\left(k+\frac{1}{2}\right) t \right) dg(t)$$

$$= O\left(\sum_{k=0}^{n} |\Delta d_{n,k}| \right) + \frac{2}{\pi} \left(\int_0^\delta + \int_\delta^\pi \right) \left(\sum_{k=0}^{n} \Delta d_{n,k} \sin\left(k+\frac{1}{2}\right) t dg(t) \right)$$

$$= 0(1) + P + Q, \text{ say,}$$

will be defined in a moment

Since $g(t)$ is of bounded variation in $(0, \pi)$ we can always find a δ in $(0, \pi)$ depending on ϵ previously chosen such that

(2.5) $$\int_0^\delta | dg(t) | < \frac{\pi}{2k} \epsilon .$$

Since in $\left| \sin \left(k + \frac{1}{2}\right) t \right| \leq 1$, we have by using (2.5) and (1 5) (*ii*)

(2.6) $$| P | < \epsilon .$$

Applying ABEL's transformation, we have

$$\sum_{k=0}^{n} \Delta d_{n,k} \sin \left(k + \frac{1}{2}\right) t = \sum_{k=0}^{n} \Delta^2 d_{n,k} \left(\sum_{r=0}^{k} \sin \left(r + \frac{1}{2}\right) t \right)$$

$$= \sum_{k=0}^{n} \Delta^2 d_{n,k} \frac{1 - \cos (k + 1)t}{2 \sin \frac{t}{2}} .$$

Hence we have for a fixed δ,

$$\max \left(\sum_{k=0}^{n} \Delta d_{n,k} \sin \left(k + \frac{1}{2}\right) t \right) \leq \frac{1}{\sin \frac{\delta}{2}} \sum_{k=0}^{n} | \Delta^2 d_{n,k} |$$

(2.7) $$= O(1) \quad \text{as} \to \infty.$$

Makinge use of (2.4) and (2.7), we obtain

(2.8) $$Q = O(1) \quad \text{as } n \to \infty.$$

Combining (2.2), (2.3), (2.4), (2.6) and (2.8) we see that

$$\sigma(n) \leq \epsilon + O(1).$$

Finally choosing ϵ first and then making $n \to \infty$, $\sigma(n) \to 0$ as $n \to \infty$. This completes the proof of the theorem.

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