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SEZIONE SCIENTIFICA

BREVI NOTE

On the absolute summability factors of double infinite series.

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Summary. - *In this paper the author has investigated the absolute summability factors of double infinite series. The theorem proved extends a result of Pati [11] to double series.*

1. DEFINITION 1. – A double infinite series $\Sigma\Sigma a_{m,n}$ [illegible] said to be absolutely summable (C, α, β) or summable [illegible] -1, if

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left| s_{m,n}^{\alpha,\beta} - s_{m-1,n}^{\alpha,\beta} - s_{m,n-1}^{\alpha,\beta} + s_{m-1,n-1}^{\alpha,\beta} \right|$$

$$= \Sigma\Sigma (mn)^{-1} \left| t_{m,n}^{\alpha,\beta} \right| < \infty,$$

$$\sum_{m=1}^{\infty} \left| s_{m,0}^{\alpha,\beta} - s_{m-1,0}^{\alpha,\beta} \right| = \Sigma\, m^{-1} \left| t_m^{\alpha} \right| < \infty,$$

$$\sum_{n=1}^{\infty} \left| s_{0,n}^{\alpha,\beta} - s_{0,n-1}^{\alpha,\beta} \right| = \Sigma\, n^{-1} \left| t_n^{\beta} \right| < \infty,$$

where $s_{m,n}^{\alpha,\beta}$ and $t_{m,n}^{\alpha,\beta}$ are the (c, α, β) mean of the sequence $\{s_{m,n}\}$ and $\{mna_{m,n}\}$ respectively and $s_{m,n}$ is the (m, n) th partial sum.

This is known [14; 15; 16].

DEFINITION 2. We have

$$t_{m,n}^{\alpha,\beta} = (A_m^{\alpha} A_n^{\beta})^{-1} \sum_{k=0}^{n} \sum_{l=0}^{m} A_{m-l}^{\alpha-1} A_{n-k}^{\beta-1} k\, l a_{l,k}$$

(*) Pervenuta alla Segreteria dell' U. M. I. il 4 Settembre 1961

$$t_m^\alpha = (A_m^\alpha)^{-1} \sum_{l=0}^{m} A_{m-l}^{\alpha-1} l a_{l,0}$$

$$\sum_{\nu=0}^{\infty} A_\nu^\rho x^\nu = (1-x)^{-\rho-1} \quad \text{for} \quad |x| < 1,$$

and by definition [8; 13],

$$A_{-1}^\alpha = 0, \quad A_0^{-1} = 1, \quad A_m^{-1} = 0 (m \geq 1), \quad A_m^{-2} = 0 (m \geq 2),$$

$$A_m^\alpha = \begin{cases} \binom{m+\alpha}{m}, & (\alpha > -1), \\ (-1)^m \binom{-\alpha-1}{m}, & (\alpha \leq -1) \end{cases}$$

$$A_m^\alpha = \frac{\Gamma(m+\alpha+1)}{\Gamma(m+1)\Gamma(\alpha+1)},$$

$$\sim \frac{m^\alpha}{\Gamma(\alpha+1)}, \qquad (\alpha \neq -1, -2, \ldots).$$

DEFINITION 3. - For any sequence $\{\varepsilon_{m,n}\}$, we write

$$\Delta^{0,0}\varepsilon_{m,n} = \varepsilon_{m,n}$$

$$\Delta^{1,1}\varepsilon_{m,n} = \varepsilon_{m+1,n+1} - \varepsilon_{m,n+1} - \varepsilon_{m+1,n} + \varepsilon_{m,n}$$

$$\Delta^{\sigma,\rho}\varepsilon_{m,n} = \sum_{\mu=0}^{\infty} \sum_{\nu=0}^{\infty} A_\mu^{-\sigma-1} A_\nu^{-\rho-1} \varepsilon_{\mu+m,\nu+n}$$

provided this series is convergent. It is easy to show that [9]

$$\Delta^{i,j}\Delta^{h,k}\varepsilon_{m,n} = \Delta^{i+h,j+k}\varepsilon_{m,n}$$

$$(1.1) \qquad \Delta^{h,k}(\delta_m \delta_n \varepsilon_{m,n}) = \sum_{\mu=0}^{h} \sum_{\nu=0}^{k} \binom{h}{\mu}\binom{k}{\nu} \Delta^\mu \delta_m \Delta^\nu \delta_n \Delta^{h-\mu,k-\nu} \varepsilon_{m+\mu,n+\nu}.$$

We also write

$$\bar{\varepsilon}_{m,n} = \Delta\varepsilon_{m,n}$$

$$E^{\alpha,\rho;\,\beta,\sigma}(M,\ N,\ m,\ n) = \sum_{\mu=0}^{M}\sum_{\nu=0}^{N} A_{N-\nu}^{\beta-1} A_{M-\mu}^{\alpha-1} A_{\mu}^{-\rho-1} A_{\nu}^{-\sigma-1} \varepsilon_{\mu+m,\nu+n}$$

$$\bar{E}^{\alpha,\rho;\,\beta,\sigma}(M,\ N,\ m,\ n) = \sum_{\mu=0}^{M}\sum_{\nu=0}^{N} A_{N-\nu}^{\beta-1} A_{M-\mu}^{\alpha-1} A_{\mu}^{-\rho-1} A_{\nu}^{-\sigma-1} \bar{\varepsilon}_{\mu+m,\nu+n}$$

$$\bar{E}_{\mu}^{\alpha,\rho;\,\beta,\sigma}(M,\ N,\ m,\ n) = \sum_{\mu=0}^{M}\sum_{\nu=0}^{N} A_{N-\nu}^{\beta-1} A_{M-\mu}^{\alpha-1} A_{\mu}^{-\rho-1} A_{\nu}^{-\sigma-1} \Delta^{1,0}\varepsilon_{\mu+m,\nu+n}$$

$$\bar{E}_{\nu}^{\alpha,\rho;\,\beta,\sigma}(M,\ N,\ m,\ n) = \sum_{\mu=0}^{M}\sum_{\nu=0}^{N} A_{N-\nu}^{\beta-1} A_{M-\mu}^{\alpha-1} A_{\mu}^{-\rho-1} A_{\nu}^{-\sigma-1} \Delta^{0,1}\varepsilon_{\mu+m,\nu+n}$$

$$T_{m,n}^{\alpha,\beta} = \sum_{\mu=1}^{m}\sum_{\nu=1}^{n} (\mu\nu)^{-1} t_{\mu,\nu}^{\alpha,\beta}$$

so that

$$E^{\alpha,\rho;\,\beta,\sigma}(-1,\ -1,\ m,\ n) = 0, \qquad E^{\alpha,\rho,\beta,\sigma}(-1,\ N,\ m,\ n) = 0,$$

$$E^{\alpha,\rho;\,\beta,\sigma}(M,\ -1,\ m,\ n) = 0, \qquad T_{0,0}^{\alpha,\beta} = 0.$$

2. In 1908 Bromwich [2] proved a theorm on convergence factors in series summable by Cesàro-means of integral order which included the various previous results. An extension of Bromwich's theorem to series summable by Cesàro means of non-integral order was given by Chapman [3] in 1911. In 1912, in connection with the summability (C, 1, 1) of the double Fourier series and its application to certain problems of mathematical physics, C. N. Moore [9] extended Bromwich's theorem on the summability factors of simple infinite series to the case of double Fourier series. In 1922, Eversull [7] gave the analogous extension to triple series.

Absolute summability factors of Fourier series were first considered by Prasad [12] in 1933. Various results concerning these investigations are due to Cheng [4], Chow [5] Pati [10] and others.

We prove the following theorem:

THEOREM. - If α, β are integers ≥ 0, and if

$$s_{m,n}^{\alpha,\beta} = O(\lambda_m \lambda_n), \quad s_{m,0}^{\alpha,\beta} = O(\lambda_m), \quad s_{0,n}^{\alpha,\beta} = O(\lambda_n),$$

where λ_m, λ_n are positive, monotonic, non-decreasing sequences, then the sufficient conditions that $\Sigma\Sigma\, a_{m,n}\varepsilon_{m,n}$ should be summable $|C, \alpha, \beta|$ are

$$\Sigma\Sigma\, \lambda_m \lambda_n \,|\, \varepsilon_{m,n} \,| < \infty,$$

$$\Sigma\Sigma\, m^{\alpha} n^{\beta} \lambda_m \lambda_n \,|\, \Delta^{\alpha+1,\beta+1} \varepsilon_{m,n} \,| < \infty,$$

$$\lambda_m \Sigma\, \lambda_n \,|\, \varepsilon_{m,n} \,| < \infty,$$

$$\lambda_n \Sigma\, \lambda_m \,|\, \varepsilon_{m,n} \,| < \infty,$$

$$m^{\alpha} \lambda_m \Sigma\, n^{\beta} \lambda_n \,|\, \Delta^{0,\beta+1} \varepsilon_{m,n} \,| < \infty,$$

$$n^{\beta} \lambda_n \Sigma\, m^{\alpha} \lambda_m \,|\, \Delta^{\alpha+1,0} \varepsilon_{m,n} \,| < \infty.$$

This extends a result of PATI [11] to double series.

3. We require tho following lemmas:

LEMMA 1. - If $\sigma > -1$, and $\sigma - \delta > 0$, then

$$\sum_{n=\nu}^{\infty} \frac{A_{n-\nu}^{\delta}}{n A_n^{\sigma}} = \sum_{n=0}^{\infty} \frac{A_n^{\delta}}{(n+\nu) A_{n+\nu}^{\sigma}} = \frac{1}{\nu A_{\nu}^{\sigma-\delta-1}}.$$

This is known [6].

LEMMA 2. - If $p > 0$, $q > 0$, and $|\lambda_m|$, $|\lambda_n|$, are the positive, monotonic, non-decreasing sequences, and

$$\Sigma\Sigma\, n^p m^q \,|\, \Delta^{q+1,p+1} \varepsilon_{m,n} \,|\, \lambda_m \lambda_n < \infty$$

then

$$\Sigma\Sigma\, n^{p_1} m^{q_1} \,|\, \Delta^{q_1+1,p_1+1} \varepsilon_{m,n} \,|\, \lambda_m \lambda_n < \infty,$$

for every p_1, q_1, such that $0 \leq p_1 \leq p$, $0 \leq q_1 \leq q$.

The proof runs parallel to lemma 2 of BOSANQUET [1].

LEMMA 3. – If $\alpha > 0$, $\beta > 0$, $\rho > 0$, $\sigma < 0$, then

$$\left.\begin{matrix} E^{\alpha,\rho;\,\beta,\sigma}(M,N,m,n) \\ \bar{E}^{\alpha,\rho;\,\beta,\sigma}(M,N,m,n) \end{matrix}\right\} = \sum_{l=0}^{h}\sum_{r=0}^{k}\binom{h}{l}\binom{k}{r}\sum_{\mu=0}^{M}\sum_{\nu=0}^{N} A_{N-\nu}^{\beta-r-1} A_{M-\mu}^{\alpha-l-1} A_{\nu}^{-\sigma+k-1}$$

$$A_{\mu}^{-\rho+h-1}\Delta^{h-l,\,k-r}\left\{\begin{matrix} \varepsilon_{\mu+m+l,\,\nu+n+r} \\ \varepsilon_{\mu+m+l,\,\nu+n+r} \end{matrix}\right.$$

This can be obtained by applying partial summation and using (1.1).

4. Proof of the theorem. Let $x_{m,n} = x_{m,n}^{0,0} = mna_{m,n}\varepsilon_{m,n}$ and let $\tau_{m,n}^{\alpha,\beta}$ denote the (m, n) th CESARO mean of order α, β, of the sequence $\{x_{m,n}\}$. By hypothesis and definition, we get

$$T_{m,n}^{\alpha,\beta} = O(\lambda_m\lambda_n) \quad \text{as} \quad (m,\ n) \to \infty,$$

$$T_{m,0}^{\alpha,\beta} = O(\lambda_m) \quad \text{as} \quad m \to \infty,$$

$$T_{0,n}^{\alpha,\beta} = O(\lambda_n) \quad \text{as} \quad n \to \infty.$$

The case $\alpha = \beta = 0$ is obvious. Let us consider the case $\alpha \geq 1$, $\beta \geq 1$. We have

$$A_m^{\alpha} A_n^{\beta} \tau_{m,n}^{\alpha,\beta} = \sum_{\mu=1}^{m}\sum_{\nu=1}^{n} A_{n-\nu}^{\beta-1} A_{m-\mu}^{\alpha-1} \mu\nu a_{\mu,\nu}\varepsilon_{\mu,\nu}$$

$$= \sum_{\mu=1}^{m}\sum_{\nu=1}^{n} A_{n-\nu}^{\beta-1} A_{m-\mu}^{\alpha-1}\varepsilon_{\mu,\nu} \sum_{l=1}^{\mu}\sum_{k=1}^{\nu} A_{\nu-k}^{\beta-1} A_{\mu-l}^{\alpha-1} A_l^{\alpha} A_k^{\beta} t_{l,k}^{\alpha,\beta}$$

$$= \sum_{l=1}^{m}\sum_{k=1}^{n} A_l^{\alpha} A_k^{\beta} t_{l,k}^{\alpha,\beta} \sum_{\mu=0}^{m-l}\sum_{\nu=0}^{n-k} A_{n-k-\nu}^{\beta-1} A_{m-l-\mu}^{\alpha-1} A_{\nu}^{-\beta-1}$$

$$A_{\mu}^{-\alpha-1}\varepsilon_{\mu+l,\,\nu+k}.$$

$$= \sum_{l=1}^{m}\sum_{k=1}^{n} A_l^{\alpha} A_k^{\beta} t_{l,k}^{\alpha,\beta} E^{\alpha,\alpha;\,\beta,\beta}(m-l,\ n-k,\ l,\ k)$$

$$= \sum_{l=1}^{m}\sum_{k=1}^{n} lk A_l^{\alpha} A_k^{\beta} E^{\alpha,\alpha;\,\beta,\beta}(m-l,\ n-k,\ l,\ k)\cdot\frac{t_{l,k}^{\alpha,\beta}}{lk}.$$

Hence by ABEL's transformation, we have

$$A_m^\alpha A_n^\beta \tau_{m,n}^{\alpha,\beta}$$

$$= \sum_{l=1}^{m-1} \sum_{k=1}^{n-1} \Delta^{l,k} \{ lkA_l^\alpha A_k^\beta E^{\alpha,\alpha;\,\beta,\beta}(m-l,\, n-k,\, l,\, k) \} T_{l,k}^{\alpha,\beta}$$

$$+ \sum_{k=1}^{n-1} \Delta^{0,k} \{ mA_m^\alpha kA_k^\beta E^{\alpha,\alpha;\,\beta,\beta}(0,\, n-k,\, m,\, k) \} T_{m,k}^{\alpha,\beta}$$

$$+ \sum_{l=1}^{m-1} \Delta^{l,0} \{ nA_n^\beta lA_l^\alpha E^{\alpha,\alpha;\,\beta,\beta}(m-l,\, 0,\, l,\, n) \} T_{l,n}^{\alpha,\beta}$$

$$+ mnA_m^\alpha A_n^\beta E^{\alpha,\alpha;\,\beta,\beta}(0,\, 0,\, m,\, n) T_{m,n}^{\alpha,\beta}$$

$$= \sum_{l=1}^{m} \sum_{k=1}^{n} \Delta^{l,k} \{ lkA_l^\alpha A_k^\beta E^{\alpha,\alpha;\,\beta,\beta}(m-l,\, n-k,\, l,\, k) \} T_{l,k}^{\alpha,\beta}$$

$$+ mA_m^\alpha \sum_{k=1}^{n} \Delta^{0,k} \{ kA_k^\beta E^{\alpha,\alpha;\,\beta,\beta}(0,\, n-k,\, m,\, k) \} T_{m,k}^{\alpha,\beta}$$

$$+ nA_n^\beta \sum_{l=1}^{m} \Delta^{l,0} \{ lA_l^\alpha E^{\alpha,\alpha;\,\beta,\beta}(m-l,\, 0,\, l,\, n) \} T_{l,n}^{\alpha,\beta}$$

$$= \sum_{l=1}^{m} \sum_{k=1}^{n} \Delta^{l,k}(lkA_l^\alpha A_k^\beta) E^{\alpha,\alpha;\,\beta,\beta}(m-l,\, n-k,\, l,\, k) T_{l,k}^{\alpha,\beta}$$

$$+ \sum_{l=1}^{m} \sum_{k=1}^{n} (l+1)(k+1) A_{k+1}^\beta A_{l+1}^\alpha \Delta^{l,k} \{ E^{\alpha,\alpha;\,\beta,\beta}(m-l,\, n-k,\, l,\, k) \} T_{l,k}^{\alpha,\beta}$$

+ other similar expressions.

Now,

$$\Delta^{l,k} E^{\alpha,\alpha;\,\beta,\beta}(m-l,\, n-k,\, l,\, k)$$

$$= \sum_{\mu=0}^{m-l} \sum_{\nu=0}^{n-k} A_{n-k-\nu}^{\beta-1} A_\nu^{-\beta-1} A_{m-l-\mu}^{\alpha-1} A_\mu^{-\alpha-1} \varepsilon_{\mu+l,\,\nu+k}$$

$$- \sum_{\mu=0}^{m-l-1} \sum_{\nu=0}^{n-k} A_{n-k-\nu}^{\beta-1} A_\nu^{-\beta-1} A_{m-l-\mu-1}^{\alpha-1} A_\mu^{-\alpha-1} \varepsilon_{\mu+l+1,\,\nu+k}$$

$$- \sum_{\mu=0}^{m-l} \sum_{\nu=0}^{n-k-1} A_{n-k-\nu-1}^{\beta-1} A_{\nu}^{-\beta-1} A_{m-l-\mu}^{\alpha-1} A_{\mu}^{-\alpha-1} \varepsilon_{\mu+l,\nu+k+1}$$

$$+ \sum_{\mu=0}^{m-l-1} \sum_{\nu=0}^{n-k-1} A_{n-k-\nu-1}^{\beta-1} A_{\nu}^{-\beta-1} A_{m-l-\mu-1}^{\alpha-1} A_{\mu}^{-\alpha-1} \varepsilon_{\mu+l+1,\nu+k+1}$$

$$= \sum_{\mu=0}^{m-l-1} \sum_{\nu=0}^{n-k-1} A_{\mu}^{-\alpha-1} A_{\nu}^{-\beta-1} \Delta^{l,k} \{ A_{n-k-\nu}^{\beta-1} A_{m-l-\mu}^{\alpha-1} \varepsilon_{\mu+l,\nu+k} \}$$

$$+ A_{n-k}^{-\beta-1} A_{m-l}^{-\alpha-1} \varepsilon_{m,n} - A_{n-k}^{-\beta-1} \sum_{\mu=0}^{m-l-1} A_{m-l-\mu-1}^{\alpha-1} A_{\mu}^{-\alpha-1} \varepsilon_{\mu+l+1,n}$$

$$- A_{m-l}^{-\alpha-1} \sum_{\nu=0}^{n-k-1} A_{n-k-\nu-1}^{\beta-1} A_{\nu}^{-\beta-1} \varepsilon_{m,\nu+k+1}$$

$$= \sum_{\mu=0}^{m-l-1} \sum_{\nu=0}^{n-k-1} A_{\mu}^{-\alpha-1} A_{\nu}^{-\beta-1} A_{m-l-\mu}^{\alpha-1} A_{n-k-\nu}^{\beta-1} \{ \Delta^{l,k} \varepsilon_{\mu+l,\nu+k} \}$$

$$+ \sum_{\mu=0}^{m-l-1} \sum_{\nu=0}^{n-k-1} A_{\mu}^{-\alpha-1} A_{\nu}^{-\beta-1} A_{n-k-\nu}^{\beta-1} A_{m-l-\mu}^{\alpha-2} \{ \Delta^{0,k} \varepsilon_{\mu+l+1,\nu+k} \}$$

$$+ \sum_{\mu=0}^{m-l-1} \sum_{\nu=0}^{n-k-1} A_{\mu}^{-\alpha-1} A_{\nu}^{-\beta-1} A_{m-l-\mu}^{\alpha-1} A_{n-k-\nu}^{\beta-2} \{ \Delta^{l,0} \varepsilon_{\mu+l,\nu+k+1} \}$$

$$+ \sum_{\mu=0}^{m-l-1} \sum_{\nu=0}^{n-k-1} A_{\mu}^{-\alpha-1} A_{\nu}^{-\beta-1} A_{m-l-\mu}^{\alpha-2} A_{n-k-\nu}^{\beta-2} \{ \varepsilon_{\mu+l+1,\nu+k+1} \}$$

$$+ A_{m-l}^{-\alpha-1} A_{n-k}^{-\beta-1} \varepsilon_{m,n} - A_{n-k}^{-\beta-1} \sum_{\mu=0}^{m-l-1} A_{m-l-\mu-1}^{\alpha-1} A_{\mu}^{-\alpha-1} \varepsilon_{\mu+l+1,n}$$

$$- A_{m-l}^{-\alpha-1} \sum_{\nu=0}^{n-k-1} A_{n-k-\nu-1}^{\beta-1} A_{\nu}^{-\beta-1} \varepsilon_{m,\nu+k+1}$$

$$= \sum_{\mu=0}^{m-l} \sum_{\nu=0}^{n-k} A_{\mu}^{-\alpha-1} A_{\nu}^{-\beta-1} A_{m-l-\mu}^{\alpha-1} A_{n-k-\nu}^{\beta-1} \{ \Delta^{l,k} \varepsilon_{\mu+l,\nu+k} \}$$

$$+ \sum_{\mu=0}^{m-l} \sum_{\nu=0}^{n-k} A_{\mu}^{-\alpha-1} A_{\nu}^{-\beta-1} A_{n-k-\nu}^{\beta-1} A_{m-l-\mu}^{\alpha-2} \{ \Delta^{0,k} \varepsilon_{\mu+l+1,\nu+k} \}$$

$$+\sum_{\mu=0}^{m-l}\sum_{\nu=0}^{n-k} A_{\mu}^{-\alpha-1}A_{\nu}^{-\beta-1}A_{n-k-\nu}^{\beta-2}A_{m-l-\mu}^{\alpha-1}\,|\,\Delta^{l,0}\epsilon_{\mu+l,\,\nu+k+1}\,|$$

$$+\sum_{\mu=0}^{m-l}\sum_{\nu=0}^{n-k} A_{\mu}^{-\alpha-1}A_{\nu}^{-\beta-1}A_{n-k-\nu}^{\beta-2}A_{m-l-\mu}^{\alpha-2}\,|\,\epsilon_{\mu+l+1,\,\nu+k+1}\,|$$

$$=\bar{E}^{\alpha,\alpha;\,\beta,\beta}(m-l,\,n-k,\,l,\,k)+\underset{k}{\bar{E}}^{\alpha,\alpha-1;\,\beta,\beta}(m-l.\,n-k,\,l+1,\,k)$$

$$+\underset{l}{\bar{E}}^{\alpha,\alpha;\,\beta,\beta-1}(m-l,\,n-k,\,l,\,k+1)+E^{\alpha,\alpha-1;\,\beta,\beta-1}(m-l,\,n-k,\,l+1,\,k+1)$$

Hence

$$A_m^{\alpha}A_n^{\beta}\tau_{m,n}^{\alpha,\beta}$$

$$=\sum_{l=1}^{m}\sum_{k=1}^{n}\Delta^{l,k}(lkA_l^{\alpha}A_k^{\beta})E^{\alpha,\alpha;\,\beta,\beta}(m-l,\,n-k,\,l,\,k)T_{l,k}^{\alpha,\beta}$$

$$+\sum_{l=1}^{m}\sum_{k=1}^{n}(l+1)(k+1)A_{l+1}^{\alpha}A_{k+1}^{\beta}\bar{E}^{\alpha,\alpha;\,\beta,\beta}(m-l,\,n-k,\,l,\,k)T_{l,k}^{\alpha,\beta}$$

$$+\sum_{l=1}^{m}\sum_{k=1}^{n}(l+1)(k+1)A_{l+1}^{\alpha}A_{k+1}^{\beta}\underset{k}{\bar{E}}^{\alpha,\alpha-1;\,\beta,\beta}(m-l,\,n-k,\,l+1,\,k)T_{l,k}^{\alpha,\beta}$$

$$+\sum_{l=1}^{m}\sum_{k=1}^{n}(l+1)(k+1)A_{l+1}^{\alpha}A_{k+1}^{\beta}\underset{l}{\bar{E}}^{\alpha,\alpha;\,\beta,\beta-1}(m-l,\,n-k,\,l,\,k+1)T_{l,k}^{\alpha,\beta}$$

$$+\sum_{l=1}^{m}\sum_{k=1}^{n}(l+1)(k+1)A_{l+1}^{\alpha}A_{k+1}^{\beta}E^{\alpha,\alpha-1;\,\beta,\beta-1}(m-l,\,n-k,\,l+1,\,k+1)T_{l,k}^{\alpha,\beta}$$

+ similar expressions.

Therefore

$$\sum_{2}^{\infty}\sum_{2}^{\infty}(mn)^{-1}\,|\,\tau_{m,n}^{\alpha,\beta}\,|$$

$$\leq\sum\sum\,(mnA_m^{\alpha}A_n^{\beta})^{-1}\,|\,\Delta^{m,n}(mnA_m^{\alpha}A_n^{\beta})\,|\,|\,\epsilon_{m,n}\,|\,|\,T_{m,n}^{\alpha,\beta}\,|$$

$$+ \Sigma\Sigma (mnA_m^\alpha A_n^\beta)^{-1}(m+1)(n+1)A_{m+1}^\alpha A_{n+1}^\beta |\Delta\varepsilon_{m,n}| \, |T_{m,n}^{\alpha,\beta}|$$

$$+ \Sigma\Sigma (mnA_m^\alpha A_n^\beta)^{-1}(m+1)(n+1)A_{m+1}^\alpha A_{n+1}^\beta |\varepsilon_{m+n,n+1}| \, |T_{m,n}^{\alpha,\beta}|$$

$$+ \Sigma\Sigma (mnA_m^\alpha A_n^\beta)^{-1} \sum_{l=1}^{m-1} \sum_{k=1}^{n-1} |\Delta^{l,k}(lkA_l^\alpha A_k^\beta)|$$

$$|E^{\alpha,\alpha;\beta,\beta}(m-l,\ n-k,\ l$$

$$+ \Sigma\Sigma (mnA_m^\alpha A_n^\beta)^{-1} \sum_{l=1}^{m-1} \sum_{k=1}^{n-1} (l+1)(k+1)A_{l+1}^\alpha A_{k+1}^\beta$$

$$|\bar{E}^{\alpha,\alpha;\beta,\beta}(m-l,\ n-k,\ l,\ k)| \, |T_{l,k}^{\alpha,\beta}|$$

$$+ \Sigma\Sigma (mnA_m^\alpha A_n^\beta)^{-1} \sum_{l=1}^{m-2} \sum_{k=1}^{n-2} (l+1)(k+1)A_{l+1}^\alpha A_{k+1}^\beta$$

$$|E^{\alpha,\alpha-1,\beta,\beta-1}(m-l,\ n-k,\ l+1,\ k+1)| \, |T_{l,k}^{\alpha,\beta}|$$

$$+ \Sigma\Sigma (mnA_m^\alpha A_n^\beta)^{-1} \sum_{l=1}^{m-1} \sum_{k=1}^{n-1} (l+1)(k+1)A_{l+1}^\alpha A_{k+1}^\beta$$

$$|\underset{k}{\bar{E}}^{\alpha,\alpha-1;\beta,\beta}(m-l,\ n-k,\ l+1,\ k)| \, |T_{l,k}^{\alpha,\beta}|$$

$$+ \Sigma\Sigma (mnA_m^\alpha A_n^\beta)^{-1} \sum_{l=1}^{m-1} \sum_{k=1}^{n-1} (l+1)(k+1)A_{l+1}^\alpha A_{k+1}^\beta$$

$$|\underset{l}{\bar{E}}^{\alpha,\alpha;\beta,\beta-1}(m-l,\ n-k,\ l,\ k+1)| \, |T_{l,k}^{\alpha,\beta}|$$

+ other terms of the same form.

Thus it suffice for our purpose to show that

(4.1) $$\Sigma\Sigma (mn)^{-1}\lambda_m\lambda_n |\varepsilon_{m,n}| < \infty,$$

(4.2) $$\Sigma\Sigma \lambda_m\lambda_n |\Delta\varepsilon_{m,n}| < \infty,$$

(4.3) $$\Sigma\Sigma \lambda_m\lambda_n |\varepsilon_{m+1,n+1}| < \infty,$$

$$(4.4)\quad \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} A_l^{\alpha}A_k^{\beta}\lambda_k\lambda_l \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{|E^{\alpha,\alpha;\,\beta,\beta}(m,n,l,k)|}{(n+k)(m+l)A_{n+k}^{\beta}A_{m+l}^{\alpha}} < \infty.$$

$$(4.5)\quad \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} A_l^{\alpha+1}A_k^{\beta+1}\lambda_k\lambda_l \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{|\bar{E}^{\alpha,\alpha;\,\beta,\beta}(m,n,l,k)|}{(n+k)(m+l)A_{n+k}^{\beta}A_{m+l}^{\alpha}} < \infty,$$

$$(4.6)\quad \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} A_l^{\alpha+1}A_k^{\beta+1}\lambda_k\lambda_l \sum_{2}^{\infty}\sum_{2}^{\infty} \frac{|E^{\alpha,\alpha-1;\,\beta,\beta-1}(m,n,l+1,k+1)|}{(n+k)(m+l)A_{n+k}^{\beta}A_{m+l}^{\alpha}} < \infty,$$

$$(4.7)\quad \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} A_l^{\alpha+1}A_k^{\beta+1}\lambda_k\lambda_l \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{\left|\frac{\bar{E}^{\alpha,\alpha;\,\beta,\beta-1}(m,n,l,k+1)}{l}\right|}{(n+k)(m+l)A_{n+k}^{\beta}A_{m+l}^{\alpha}} < \infty,$$

$$(4.8)\quad \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} A_l^{\alpha+1}A_k^{\beta+1}\lambda_k\lambda_l \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{\left|\frac{\bar{E}^{\alpha,\alpha-1,\,\beta,\beta}(m,n,l+1,k)}{k}\right|}{(n+k)(m+l)A_{n+k}^{\beta}A_{m+l}^{\alpha}} < \infty.$$

The inequalities from (4.1) to (4.3) follow from the hypothesis. Out of the remaining, we shall consider the typical ones.

By lemma 3, since $m \geq 1$, $n \geq 1$, $0 \leq r \leq \alpha - 1$, $0 \leq \delta \leq \beta - 1$, we have

$$\sum_{l=1}^{\infty}\sum_{k=1}^{\infty} A_l^{\alpha}A_k^{\beta}\lambda_k\lambda_l \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{A_m^{\alpha-r-1}A_n^{\beta-\delta-1}\,|\Delta^{\alpha-r,\,\beta-\delta}\varepsilon_{l+r,\,k+\delta}|}{(n+k)(m+l)A_{n+k}^{\beta}A_{m+l}^{\alpha}}$$

$$\leq \sum_{l=1}^{\infty}\sum_{k=1}^{\infty} A_l^{\alpha}A_k^{\beta}\lambda_k\lambda_l\,|\Delta^{\alpha-r,\,\beta-\delta}\varepsilon_{l+r,\,k+\delta}| \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{A_m^{\alpha-r-1}A_n^{\beta-\delta-1}}{(n+k)(m+l)A_{n+k}^{\beta}A_{m+l}^{\alpha}}$$

$$= \sum_{l=1}^{\infty}\sum_{k=1}^{\infty} A_l^{\alpha}A_k^{\beta}\lambda_k\lambda_l\,|\Delta^{\alpha-r,\,\beta-\delta}\varepsilon_{l+r,\,k+\delta}|\,\frac{1}{kA_k^{\delta}lA_l^{r}}$$

$$\leq K\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} k^{\beta-\delta-1}l^{\alpha-r-1}\lambda_k\lambda_l\,|\Delta^{\alpha-r,\,\beta-\delta}\varepsilon_{l+r,\,k+\delta}| < \infty,$$

by hypothesis and lemma 2. This proves (4.4).

Again

$$\sum_{l=1}^{\infty} \sum_{k=1}^{\infty} A_l^{\alpha+1} A_k^{\beta+1} \lambda_k \lambda_l \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{A_m^{\alpha-r-1} A_n^{\beta-\delta-1} |\Delta^{\alpha-r,\,\beta-\delta} \varepsilon_{l+r,\,k+\delta}|}{(m+l)(n+k) A_{m+l}^{\alpha} A_{n+k}^{\beta}}$$

$$\leq \sum_{l=1}^{\infty} \sum_{k=1}^{\infty} A_l^{\alpha+1} A_k^{\beta+1} \lambda_k \lambda_l \, |\Delta^{\alpha-r+1,\,\beta-\delta+1} \varepsilon_{l+r,\,k+\delta}| \sum_{0}^{\infty} \sum_{0}^{\infty} \frac{A_m^{\alpha-r-1} A_n^{\beta-\delta-1}}{(n+k)(m+l) A_{n+k}^{\beta} A_{m+l}^{\alpha}}$$

$$= \sum_{l=1}^{\infty} \sum_{k=1}^{\infty} A_l^{\alpha+1} A_k^{\beta+1} \lambda_k \lambda_l \, |\Delta^{\alpha-r+1,\,\beta-\delta+1} \varepsilon_{l+r,\,k+\delta}| \frac{1}{k A_k^{\delta} l A_l^{r}}$$

$$\leq K \sum_{l=1}^{\infty} \sum_{k=1}^{\infty} k^{\beta-\delta} l^{\alpha-r} \lambda_k \lambda_l \, |\Delta^{\alpha-r+1,\,\beta-\delta+1} \varepsilon_{l+r,\,k+\delta}| < \infty,$$

by hypothesis lemma 1 and lemma 2. This proves (4.5).

Also, since $n \geq 2$, $m \geq 2$, then for $0 \leq r \leq \alpha - 2$, $0 \leq \delta \leq \beta - 2$, we have

$$\sum_{l=1}^{\infty} \sum_{k=1}^{\infty} A_l^{\alpha+1} A_k^{\beta+1} \lambda_k \lambda_l \sum_{m=2}^{\infty} \sum_{n=2}^{\infty} \frac{A_m^{\alpha-r-2} A_n^{\beta-\delta-2} |\Delta^{\alpha-r,\,\beta-\delta} \varepsilon_{l+r+1,\,k+\delta+1}|}{(n+k)(m+l) A_{n+k}^{\beta} A_{m+l}^{\alpha}}$$

$$\leq \sum_{l=1}^{\infty} \sum_{k=1}^{\infty} A_l^{\alpha+1} A_k^{\beta+1} \lambda_k \lambda_l \, |\Delta^{\alpha-r,\,\beta-\delta} \varepsilon_{l+r+1,\,k+\delta+1}| \sum_{0}^{\infty} \sum_{0}^{\infty} \frac{A_n^{\beta-\delta-2} A_m^{\alpha-r-2}}{(n+k)(m+l) A_{n+k}^{\beta} A_{m+l}^{\alpha}}$$

$$= \sum_{l=1}^{\infty} \sum_{k=1}^{\infty} A_l^{\alpha+1} A_k^{\beta+1} \lambda_k \lambda_l \, |\Delta^{\alpha-r,\,\beta-\delta} \varepsilon_{l+r+1,\,k+\delta+1}| \frac{1}{k A_k^{\delta+1} l A_l^{r+1}}$$

$$\leq K \sum_{l=1}^{\infty} \sum_{k=1}^{\infty} l^{\alpha-r-1} k^{\beta-\delta-1} \lambda_k \lambda_l \, |\Delta^{\alpha-r,\,\beta-\delta} \varepsilon_{l+r+1,\,k+\delta+1}| < \infty,$$

by hypothesis, lemma 1 and lemma 2. This proves (4.6).

The remaining inequalities will follow in the same way. Thus

$$\Sigma\, \Sigma\, (mn)^{-1} \, |t_{m,n}^{\alpha,\beta}| < \infty.$$

Similarly,

$$\Sigma m^{-1} | t_m^{\alpha} | < \infty ; \qquad \Sigma n^{-1} | t_n^{\beta} | < \infty.$$

This completes the proof of the theorem

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