Bollettino
Unione Matematica Italiana

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On the uniform Riemann summability.

Bollettino dell'Unione Matematica Italiana, Serie 3, Vol. **16** (1961), n.3, p. 295–300.

Zanichelli

<http://www.bdim.eu/item?id=BUMI_1961_3_16_3_295_0>

Bollettino dell'Unione Matematica Italiana, Zanichelli, 1961.

On the uniform Riemann summability.

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Summary. - *Szasz has introduced the notion of the uniform summability* (R', 1) *of trigonometrical series. Yano generalised this result on the uniform summability* (R', 1). *In this paper, the author has extended the earlier result to uniform summability* (R', p).

1. The series $\sum_{\nu=1}^{\infty} a_\nu$ is said to be uniformly summable (R', p) to the sum s if

$$F(t) = t \sum_{\nu=1}^{n} s_\nu \left(\frac{\operatorname{Sin} \nu t}{\nu t}\right)^p \tag{1.1}$$

converges uniformly to the sum s in the interval $0 \leq t \leq \pi$, where s_n is the n^{th} partial of the series considered and p is an integer (finite) such that $p \geq 1$.

SZASZ [1] introduced the notion of uniform summability $(R', 1)$ and proved the following theorem:

THEOREM S. - If

$$\sum_{n}^{2n} (\mid a_\nu \mid - a_\nu) = O(1) \tag{1.2}$$

then the convergence of the series $\sum_{\nu=1}^{\infty} a_\nu$ implies the uniform summability $(R', 1)$ to the sum s in the interval $0 \leq t \leq \pi$.

This result was extended by YANO [2] for uniform summability $(R', 1)$. The object here is to prove the following theorem:

THEOREM I. - *If*

$$\sum_{n}^{2n} (\mid a_\nu \mid - a_\nu) = O(1) \tag{1.3}$$

(*) Pervenuta alla Segreteria dell'U. M. I. il 19 agosto 1961.

then the convergenge of the series $\sum_{\nu=1}^{\infty} a_\nu$ *implies its uniform summability* (R′, p) *to the sum* s *in the interval* $0 \leq t \leq \pi$ *for all integral (finite) values of* $p \geq 1$.

2. We shall require the following lemmas.

LEMMA 1. - If we write

$$\Phi(t) = \left(\frac{\text{Sin } t}{t}\right)^p \tag{2.1}$$

then

$$\left(\frac{d}{dt}\right)^k \Phi(t) = \Phi^{(k)}(t) = O(1) \text{ as } t \to 0 \tag{2.2}$$

and

$$\Phi^{(k)}(t) = 0(t^{-p}) \text{ as } t \to \infty . \tag{2.3}$$

for all $k = 0, 1, 2 ...$

This is due to KANNO [3].

LEMMA 2. - If

$$\sum_{n}^{2n} (\mid a_\nu \mid - a_\nu) = O(1) \tag{2.4}$$

then

$$s_n \equiv \sum_{\nu=1}^{n} a_\nu = O(1) \tag{2.5}$$

$$\sum_{n}^{2n} \mid a_\nu \mid = O(1) \tag{2.6}$$

and

$$\sum_{n}^{\infty} \nu^{-1} \mid a_\nu \mid = O(n^{-1}). \tag{2.7}$$

The lemma is due to SZASZ [4].

LEMMA 3. - If the series

$$t \sum_{\nu=1}^{\infty} s_\nu \Phi(\nu t) \tag{2.8}$$

converges in the interval $0 < t \leq \pi$, then

$$t \sum_{\nu=1}^{\infty} s_\nu \Phi(\nu t) = \sum_{1}^{\infty} a_\nu \rho_\nu(t) \tag{2.9}$$

where

$$\rho_n(t) = t \sum_{\nu=n}^{\infty} \Phi(\nu t). \tag{2.10}$$

Proof of Lemma 3. - Applying the method of partial summation, we have

$$t \sum_{n}^{m} s_\nu \Phi(\nu t) = s_n \rho_n(t) - s_m \rho_m(t) + \sum_{n+1}^{m} a_\nu \rho_\nu(t). \tag{2.11}$$

We write

$$T_n(t) = \sum_{\nu=1}^{n} \nu t \Phi(\nu t). \tag{2.12}$$

Now, again by applying partial summation to (2. 10) we get

$$\rho_n(t) = -\frac{T_n(t)}{n} + \sum_{n}^{\infty} T_\nu(t)/[\nu(\nu+1)] \tag{2.13}$$

by virtue of (2. 12).

Now

$$T_n(t) = \sum_{\nu=1}^{n} \nu t \Phi(\nu t)$$

$$= O(t) \sum_{\nu=1}^{n} \nu \cdot O(1), \quad \text{for all} \quad 0 < t < \frac{1}{n} < \frac{1}{n_0}$$

$$= O(t) \cdot O(n^2)$$

$$= O\left(\frac{1}{t}\right) \cdot O(n^2)(t^2)$$

$$= O\left(\frac{1}{t}\right). \tag{2.14}$$

Therefore

$$\rho_m(t) = -\frac{T_m(t)}{m} + \sum_{m+1}^{\infty} T_\nu(t)/[\nu(\nu+1)]$$

$$(2.15) \qquad = O\left(\frac{1}{mt}\right), \qquad \text{by (2.14).}$$

Since by Lemma 2. we have $s_m = O(1)$, hence $[s_m/m] \to 0$ as $m \to \infty$. We find that $s_m\rho_m(t) \to 0$ as $m \to 0$ for all values of $t>0$ and hence

$$\sum_{n+1}^{\infty} a_\nu \rho_\nu(t)$$

converges, by virtue of (2.11), since the series (2.8) is supposed to be convergent.

Thus from (2.11) we have

$$(2.16) \qquad t\sum_{n}^{m} s_\nu \Phi(\nu t) = s_n\rho_n + \sum_{n+1}^{m} a_\nu \rho_\nu(t),$$

as $m \to \infty$, which yields (2.9) for $n = 1$.

This completes the proof of Lemma 3.

3. Proof of Theorem 1. We write

$$t\sum_{1}^{\infty} s_\nu \Phi(\nu t) = t\left(\sum_{1}^{n} + \sum_{n+1}^{\infty}\right) s_\nu \Phi(\nu t)$$

$$(3.1) \qquad = T_1 + T_2, \qquad \text{say.}$$

Now from ABEL'S partial summation we have

$$(3.2) \qquad T_1 = tS_n\Phi(nt) + t\sum_{1}^{n-1} S_\nu \Delta_\nu \Phi(\nu t).$$

Since the series $\sum_{\nu=1}^{\infty} a_\nu$ is supposed to be convergent hence we replace a_1 by $a_1 - s$ and now we find that

$$s_n \to 0 \quad \text{as} \quad n \to \infty.$$

Therefore for a given ε such that $0<\varepsilon<\frac{1}{2}$ we choose $n_0(\varepsilon)$ such that

$$s_n = O(\varepsilon^2) \tag{3.3}$$

for all $n > n_0(\varepsilon)$.

Hence

$$\sum_{\nu=1}^{n} s_\nu \equiv S_n = O(n\varepsilon^2) \tag{3.4}$$

for all $n > n_0(\varepsilon)$.

After ε being chosen, we now suppose t to be such that $n = 1 + (\varepsilon t)^{-1}$ for $0 < t < \frac{1}{n_0}$.

In this case we have

$$n > (\varepsilon t)^{-1}. \tag{3.5}$$

Hence

$$T_1 = O(\varepsilon^2 t) + O(t) \sum_{1}^{n} \nu^{-1} \mid S_\nu \mid, \qquad \text{by Lemma 1.}$$

$$= O(\varepsilon^2) + O(nt\varepsilon^2), \qquad \text{by (3.4)}$$

$$= O(\varepsilon^2) + O(\varepsilon^2)\left(t + \frac{1}{\varepsilon}\right)$$

$$= O(\varepsilon^2) + O(\varepsilon^2 t) + O(\varepsilon)$$

$$= O(\varepsilon) + O(\varepsilon^2). \tag{3.6}$$

Considering T_2, we write

$$T_2 = t\left(\sum_{n+1}^{\lambda n} + \sum_{\lambda n+1}^{\infty}\right) s_\nu \Phi(\nu t) \tag{3.7}$$

By Lemma 4, we obtain

$$t \sum_{\lambda n+1}^{\infty} s_\nu \Phi(\nu t) = s_{\lambda n+1}\rho_{\lambda n+1}(t) + \sum_{\lambda n+2}^{\infty} a_\nu \rho_\nu(t)$$

$$= O\left(\frac{1}{\lambda n t}\right) + O\left(\frac{1}{t}\right) \sum_{\lambda n+2}^{\infty} \frac{|a_\nu|}{\nu}$$

by Lemmas 2 and 3.

$$= O\left(\frac{1}{\lambda n t}\right). \tag{3.8}$$

Since for a given $\varepsilon > 0$, we have choosen $n_0(\varepsilon)$ such that $s_n = O(\varepsilon^2)$ for all $n > n_0(\varepsilon)$, we now define $k = k(\lambda)$ by $n + k = [\lambda n]$, so that $k \leq (\lambda - 1)n$. Now for all $0 < t \leq \pi$, we assume that $\lambda = 1 + (\varepsilon n t)^{-1}$ and hence

$$\begin{aligned} t \sum_{n+}^{\lambda n} s_\nu \Phi(\nu t) &= O(t) \sum_{n+1}^{\lambda n} | s_\nu | \\ &= O(t\varepsilon^2 k) \\ &= O(t\varepsilon^2(\lambda - 1)n) \\ &= O(\varepsilon). \end{aligned} \tag{3.9}$$

And since $\lambda = 1 + (\varepsilon n t)^{-1}$ i.e. $\lambda > (\varepsilon n t)^{-1}$, hence from (3.6), (3.7), (3.8) and (3.9) we have

$$T_1 + T_2 = O(\varepsilon) + O(\varepsilon^2) \tag{3.10}$$

whic uniformly tends zero as $n > n_0(\varepsilon)$ in the whole inerval $0 < t \leq \pi$, since ε being arbitrary small and since the terms of the series (1.1) are continuous near $t = 0$, the result follows for the chosed interval $0 \leq t \leq \pi$.

This completes the proof of the Theorem 1.

I am much indebted to Professor M. L. Misra for his suggestions and criticism during the preparation of this paper.

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