A $q$-version of the Newton interpolation formula and some Eulerian identities.


Zanichelli

<http://www.bdim.eu/item?id=BUMI_1961_3_16_3_285_0>
A q-version of the Newton interpolation formula
and some eulerian identities

Nota di Richard Bellman (*)

Summary. - Using an extension of the Newton interpolation formula, it is shown how to obtain series expansions for functions such as

\[ \prod_{n=0}^{\infty} (1 + x q^n). \]

1. Introduction.

By means of the functional equation

\[ f(x) = (1 + xq)f(xq), \]

it is easy to derive the identity

\[ f(x) = \prod_{n=0}^{\infty} (1 + xq^n) = 1 + \sum_{n=1}^{\infty} \frac{x^n q^{n(n+1)/2}}{(1-q)(1-q^2) \cdots (1-q^n)}. \]

This technique, introduced by Euler, was extensively developed by Gauss and Jacobi. In a recent paper, [1], we showed how results of this nature could be obtained by means of partial fraction expansions. In this paper, we wish to present a third method for obtaining results such as (2), based upon an extension of Newton's interpolation formula.

2. q-version of Newton Formula

The classical interpolation formula of Newton has the form

\[ f(n) = a_0 + a_1 n + a_2 \frac{n(n-1)}{2!} + a_3 \frac{n(n-1)(n-2)}{3!} + \ldots, \]

where, as is easily established,

(*) Prevenuta alla Segreteria dell'U.M.I. il 26/7/61.
(2) \[ a_0 = f(0), \]
\[ a_1 = \Delta f(0) = f(1) - f(0), \]
\[ a_2 = \Delta^2 f(0) = f(2) - 2f(1) + f(0), \]
and so on.

Let us now consider an extension of this of the form

\[ f(n) = a_0 + a_1 \left( \frac{q^n - 1}{q - 1} \right) + a_2 \left( \frac{q^n - 1)(q^{n-1} - 1)}{(q - 1)(q^2 - 1)} \right) + \ldots, \]
valid for \( n = 0, 1, 2, \ldots \), To determine the coefficients, we write

(4) \[ f(n + 1) = a_0 + a_1 \left( \frac{q^{n+1} - 1}{q - 1} \right) + a_2 \left( \frac{(q^{n+1} - 1)(q^n - 1)}{(q - 1)(q^2 - 1)} \right) + \ldots, \]

and subtract, obtaining

(5) \[ \frac{f(n + 1) - f(n)}{q^n} = a_1 + \frac{a_2}{q} \left( \frac{q^n - 1}{q - 1} \right) + \frac{a_2}{q^2} \left( \frac{(q^n - 1)(q^{n-1} - 1)}{(q - 1)(q^2 - 1)} \right) + \ldots, \]

Hence, if we write

(6) \[ \Delta f(n) = \frac{f(n + 1) - f(n)}{q^n}, \]

we can write

(7) \[ \frac{a_r}{q^{r(r-1)z}} = \Delta^r f(n) \mid _{n=0}. \]

3. Application

Write

(1) \[ f(n) = \prod_{k=0}^{n} (1 + xq^k). \]
Using the foregoing results, we obtain the identity

\[
\Pi_{k=1}^{n} (1 + xq^{k}) = 1 + qx \left( \frac{q^{n} - 1}{q - 1} \right) + q^{2}x^{2} \left( \frac{q^{n-1} - 1}{q - 1} \right) + \ldots
\]

Take \( |q| < 1 \) and let \( n \to \infty \). We obtain as the limit of (2) the relation of (1.2).

4. Extensions

In a similar fashion, we can derive many other interesting identities. Furthermore, the result can be extended to functions of several variables. We can write

\[
f(m, n) = \sum_{k,l} a_{k,l} \frac{(q_{1}^{m} - 1)}{(q_{1} - 1)} \frac{(q_{1}^{m-1} - 1)}{(q_{1}^{m} - 1)} \ldots \frac{(q_{1}^{m-k+l} - 1)}{(q_{1}^{m} - 1)} \frac{(q_{2}^{n} - 1)}{(q_{2} - 1)} \frac{(q_{2}^{n-1} - 1)}{(q_{2}^{n} - 1)} \ldots \frac{(q_{2}^{n-l+1} - 1)}{(q_{2}^{n} - 1)},
\]

and thereby derive corresponding results for the function

\[
\Pi_{k,l} (1 + xq_{1}^{k}q_{2}^{l}).
\]

Finally, let us note, we can derive further results by expanding

\[
f(n) = \Pi_{k=0}^{n} (1 + xq_{1}^{k}) = 1 + q_{1} \left( \frac{q^{n} - 1}{q - 1} \right) + q_{2} \frac{(q^{n} - 1)(q^{n-1} - 1)}{(q - 1)(q^{2} - 1)}
\]

where \( q \) does not necessarily equal \( q_{1} \).

REFERENCE