BOLLETTINO
UNIONE MATEMATICA ITALIANA

RICHARD BELLMAN

A q-version of the Newton interpolation formula and some Eulerian identities.

Bollettino dell'Unione Matematica Italiana, Serie 3, Vol. **16** (1961), n.3, p. 285–287.

Zanichelli

<http://www.bdim.eu/item?id=BUMI_1961_3_16_3_285_0>

Bollettino dell'Unione Matematica Italiana, Zanichelli, 1961.

A q-version of the Newton interpolation formula and some eulerian identities

Nota di RICHARD BELLMAN (*)

Summary. - *Using an extension of the* NEWTON *interpolation formula, it is shown how to obtain series expansions for functions such as*

$$\prod_{n=0}^{\infty} (1 + x\, q^n).$$

1. Introduction.

By means of the functional equation

$$f(x) = (1 + xq)f(xq), \tag{1}$$

it is easy to derive the identity

$$f(x) = \prod_{n=0}^{\infty} (1 + xq^n) = 1 + \sum_{n=1}^{\infty} \frac{x^n q^{n(n+1)/2}}{(1-q)(1-q^2)\dots(1-q^n)}. \tag{2}$$

This technique, introduced by EULER, was extensively developed by GAUSS and JACOBI. In a recent paper, [1], we showed how results of this nature could be obtained by means of partial fraction expansions. In this paper, we wish to present a third method for obtaining results such as (2), based upon an extension of Newton's interpolation formula.

2. q-version of Newton Formula

The classical interpolation formula of NEWTON has the form

$$f(n) = a_0 + a_1 n + a_2 \frac{n(n-1)}{2!} + a_3 \frac{n(n-1)(n-2)}{3!} + \dots, \tag{1}$$

where, as is easily established,

(*) Prevenuta alla Segreteria dell'U.M.I. il 26/7/61.

$$(2)\qquad \begin{aligned} a_0 &= f(0), \\ a_1 &= \Delta f(0) = f(1) - f(0), \\ a_2 &= \Delta^2 f(0) = f(2) - 2f(1) + (0), \end{aligned}$$

and so on.

Let us now consider an extension of this of the form

$$(3)\qquad f(n) = a_0 + a_1 \frac{(q^n - 1)}{(q - 1)} + a_2 \frac{(q^n - 1)(q^{n-1} - 1)}{(q - 1)(q^2 - 1)} + \dots,$$

valid for $n = 0, 1, 2, \dots$, To determine the coefficients, we write

$$(4)\qquad f(n + 1) = a_0 + a_1 \left(\frac{q^{n+1} - 1}{q - 1}\right) + a_2 \frac{(q^{n+1} - 1)(q^n - 1)}{(q - 1)(q^2 - 1)} + \dots,$$

and subtract, obtaining

$$(5)\qquad \frac{f(n + 1) - f(n)}{q^n} = a_1 + \frac{a_2}{q} \frac{(q^n - 1)}{(- 1)} + \frac{a_3}{q^2} \frac{(q^n - 1)(q^{n-1} - 1)}{(q - 1)(q^2 - 1)} + \dots,$$

Hence, if we write

$$(6)\qquad \Delta f(n) = \frac{f(n + 1) - f(n),}{q^n}$$

we can write

$$(7)\qquad \frac{a_r}{q^{r(r-1)/2}} = \Delta^r f(n) \mid_{n=0}.$$

3. Application

Write

$$(1)\qquad f(n) = \prod_{k=0}^{n} (1 + xq^k).$$

Using the foregoing results, we obtain the identity

$$(2)\qquad \prod_{k=1}^{n}(1+xq^k)=1+qx\left(\frac{q^n-1}{q-1}\right)$$
$$+\,q^3x^2\left(\frac{q^n-1}{q-1}\right)\left(\frac{q^{n-1}-1}{q^2-1}\right)+\dots$$

Take $|q|<1$ and let $n\to\infty$. We obtain as the limit of (2) the relation of (1.2).

4. Extensions

In a similar fashion, we can derive many other interesting identities. Furthemore, the result can be extended to functions of several variables. We can write

$$(1)\qquad f(m,n)=\sum_{k,l}a_{kl}\left(\frac{q_1^m-1}{q_1-1}\right)\left(\frac{q_1^{m-1}-1}{q_1^2-1}\right)\dots\left(\frac{q_1^{m-k+}-1}{q_1^k-1}\right)$$
$$\left(\frac{q_2^n-1}{q_2-1}\right)\left(\frac{q_2^{n-1}-1}{q_2^2-1}\right)\dots\left(\frac{q_2^{n-l+1}-1}{q_2^l-1}\right),$$

and thereby derive corresponding results for the function

$$(2)\qquad \prod_{k,l}^{\infty}(1+xq_1^kq_2^l).$$

Finally, let us note, we can derive further results by expanding

$$(3)\qquad f(n)=\prod_{k=0}^{n}(1+xq_1^k)=1\,a_1\left(\frac{q^n-1}{q-1}\right)+a_2\,\frac{(q^n-1)(q^{n-1}-1)}{(q-1)(q^2-1)}$$

where q does not necessarily equal q_1.

REFERENCE

[1] R. BELLMAN. *The expansions of some infinite products*, « Duke Math J. », Vol. 24, 1957, pp. 353-356.