Bollettino
Unione Matematica Italiana

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On some results involving “associated Legendre’s” functions.

Bollettino dell’Unione Matematica Italiana, Serie 3, Vol. **16** (1961), n.3, p. 218–220.

Zanichelli

<http://www.bdim.eu/item?id=BUMI_1961_3_16_3_218_0>

On some results involving « associated Legendre's » functions.

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Summary. - *The integral and other properties, of associated Legendre's functions, have been considered.*

BLOH, E. L. [1] proved that

$$\exp(tz)I_m\left(t(z^2-1)^{\frac{1}{2}}\right)=\sum_{n=0}^{\alpha}\frac{t^{m+n}P_{n+m}^{m}(z)}{(2m+n)!}. \tag{1}$$

Putting $z = \operatorname{Cosh}\alpha$, $t = \gamma \operatorname{Sinh}\beta$ in (1) we have

$$\exp(\gamma \operatorname{Sinh}\beta \operatorname{Cosh}\alpha)\, I_m(\gamma \operatorname{Sinh}\beta \operatorname{Sinh}\alpha) = \sum_{n=0}^{\alpha}\frac{\gamma^{m+n}\operatorname{Sinh}^{m+n}\beta\, P_{m+n}^{m}(\operatorname{Cosh}\alpha)}{(2m+n)!}. \tag{2}$$

Again putting $z = \operatorname{Cosh}\beta$, $t = \gamma \operatorname{Sinh}\alpha$, we have

$$\exp(\gamma \operatorname{Sinh}\alpha \operatorname{Cosh}\beta)\, I_m(\gamma \operatorname{Sinh}\beta \operatorname{Sinh}\alpha) = \sum_{n=0}^{\alpha}\frac{\gamma^{m+n}\operatorname{Sinh}^{m+n}\alpha\, P_{m+n}^{m}(\operatorname{Cosh}\beta)}{(2m+n)!}. \tag{3}$$

Dividing (2) by (3) and simplifying we have,

$$\exp(\gamma \operatorname{Sinh}(\beta-\alpha))\sum_{n=0}^{\alpha}\frac{\gamma^{m+n}\operatorname{Sinh}^{m+n}\alpha\, P_{m+n}^{m}(\operatorname{Cosh}\beta}{(m+n)!} = \sum_{n=0}^{\alpha}\frac{\gamma^{m+u}\operatorname{Sinh}^{m+n}\beta\, P_{m+n}^{m}(\operatorname{Cosh}\alpha)}{(2m+n)!}. \tag{4}$$

(*) Pervenuta alla Segreteria dell'U. M. I. il 12 giugno 1961.

Equating the coefficients of $\frac{\gamma^{m+n}}{(2m+n)!}$ from both the sides of (4) we have

$$\text{(5)} \qquad P^m_{m+} \,(\text{Cosh}\,\alpha) = \left(\frac{\text{Sinh}\,\alpha}{\text{Sinh}\,\beta}\right)^{m+n} \sum_{\gamma'=0}^{n} P^m_{m+n-\gamma'}\,(\text{Cosh}\,\beta)$$

$$\times \left\{\begin{matrix} 2m+n \\ \gamma' \end{matrix}\right\} \left[\frac{\text{Sinh}\,(\beta-\alpha')}{\text{Sinh}\,\alpha}\right]^{\gamma'}$$

putting $i\alpha$ for α and $i\beta$ for β we have from (5)

$$\text{(6)} \qquad P^m_{m+n}\,(\text{Cos}\,\alpha) = \left(\frac{\text{Sin}\,\alpha}{\text{Sin}\,\beta}\right)^{m+n} \sum_{\gamma'=0}^{n} P^m_{m+n-\gamma'}\,(\text{Cos}\,\beta)$$

$$\times \binom{2m+n}{\gamma'} \left[\frac{\text{Sin}\,(\beta-\alpha)}{\text{Sin}\,\alpha}\right]^{\gamma'}.$$

Putting $m = 0$ in (6) we get RAINVILLE's result (3) putting $\beta = 2\alpha$ and $\text{Cos}\,2\alpha = x$ in (6) we have

$$\text{(7)} \qquad 2^{\frac{m+n}{2}}(1+x)^{\frac{m+n}{2}} P^m_{m+n}\left(\sqrt{\frac{1+x}{2}}\right)$$

$$= \sum_{\gamma'=0}^{n} \binom{2m+n}{\gamma'} P^m_{m+n-\gamma'}\,(x).$$

Hence

$$\text{(8)} \qquad \int_{-1}^{1} 2^{\frac{m+n}{2}}(1+x)^{\frac{m+n}{2}} P^m_{n+m}\left(\sqrt{\frac{1+x}{2}}\right) P^m_{s+m}(x)dx$$

$$= 0 \text{ if } s > n$$

$$= \binom{2m+n}{s} \frac{2}{2m+2s+1} \frac{(s+2m)!}{s!}$$

if $\qquad o \leq s \leq n$

$$(9)\qquad \int_{-1}^{1} 2^{\frac{m+n}{2}}(1+x)^{\frac{m+n}{2}} P_{n+m}^{m}\left(\sqrt{\frac{1+x}{2}}\right) Q_{l+m}^{m}(x)dx$$

$$= \sum_{\gamma'=0}^{n} \left\{ \begin{matrix} 2m+n \\ \gamma' \end{matrix} \right\} \int_{-1}^{1} Q_{l+m}^{m}(x) P_{m+n-\gamma'}^{m}(x)dx$$

$$= \sum_{\gamma'=0}^{n} \frac{\binom{2m+n}{\gamma'}(-1)^{m}[1-(-1)^{l+n-\gamma'}](l+2m)!}{(l-n+\gamma')(l+n-\gamma'+2m+1)l!}$$

if $l > n$

$$\int_{0}^{1} x^{\sigma}(1-x^{2})^{\frac{m}{2}} 2^{\frac{m+n}{2}}(1+x)^{\frac{m+n}{2}} P_{n+m}^{m}\left(\sqrt{\frac{1+x}{2}}\right) dx$$

$$= \sum_{\gamma'=0}^{n} \left\{ \begin{matrix} 2m+n \\ \gamma' \end{matrix} \right\} \int_{-1}^{1} x^{\sigma}(1-x^{2})^{\frac{m}{2}} P_{n+m-\gamma'}^{m}(x)dx$$

$$= \sum_{\gamma'=0}^{n} \frac{\left\{ \begin{matrix} 2m+n \\ \gamma' \end{matrix} \right\}(-1)^{m}\Gamma\left(\frac{1}{2}+\frac{\sigma}{2}\right)\Gamma\left(1+\frac{\sigma}{2}\right)\Gamma(1+2m+n-\gamma')}{2^{m+1}\Gamma(1+n-\gamma')\Gamma\left(1+\frac{\sigma}{2}-\frac{n-r'}{2}\right)\Gamma\left(\frac{3}{2}+\frac{\sigma}{2}+m+\frac{n-r}{2}\right)}$$

if Re > -1 and m a positive integer

by using the formulae Erdeli [3]

$$\int_{-1}^{1} Q_{n}^{m}(x) P_{l}^{m}(x)dx = \frac{(-1)^{m}[(1-(-1)^{1+u}](n+m)!}{(l-n)(l+n+1)(n-m)!}$$

and

$$(-1)^{m} 2^{m+1}\Gamma(1-m+\nu) \int_{0}^{1} x^{\sigma}(1-x^{2})^{\frac{m}{2}} P_{\nu}^{m}(x)dx$$

$$= \frac{\Gamma\left(\frac{1}{2}+\frac{\sigma}{2}\right)\Gamma\left(1+\frac{\sigma}{2}\right)\Gamma(1+m+\nu)}{\Gamma\left(1+\frac{\sigma}{2}+\frac{m}{2}-\frac{\nu}{2}\right)\Gamma\left(\frac{3}{2}+\frac{\sigma}{2}+\frac{m}{2}+\frac{\nu}{2}\right)}$$

if Re $\sigma > -1$, m is a positive integer.

REFERENCES

[1] Bloh E. L., « Prikl. Mat. Meh. », 18, 745-48, 1954 (Russian).
[2] Rainville E. D., « Bull. American Heth. Soc. », Vol. 51, 268-71 (1945).
[3] Erdeli A., *Higher Transcendental functions*, Vol. 1, 171-172 (1953).