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SEZIONE SCIENTIFICA

BREVI NOTE

On the Cesàro summability of the ultraspherical series.

By B. C. SINGHAI (Sagar, India) (*)

Summary. - *The author has obtained theorems for Cesàro summability of the ultraspherical series which extend and generalize the results of wang [6 and 7] of Fourier series.*

1. Let $f(\theta, \varphi)$ be a function defined for the range $0 \leq \theta \leq \pi$ and $0 \leq \varphi \leq 2\pi$, the ultraspherical series corresponding to it on the sphere S is

$$(1.1) \qquad f(\theta, \varphi) \sim \frac{1}{\pi^2} \sum_{n=0}^{\infty} (n + \lambda) \iint_S \frac{f(\theta', \varphi') P_n^{(\lambda)}(\cos \omega) \sin \theta' d\theta' d\varphi'}{[\sin^2\theta' \sin^2 (\varphi - \varphi')]^{\frac{1}{2} - \lambda}},$$

where

$$\cos \omega = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos (\varphi - \varphi').$$

The LAPLACE series is a particular case of (1.1) for $\lambda = \frac{1}{2}$ [4]. We write [see also; 5]

$$(1.2) \qquad f(\omega) = \frac{1}{2\pi(\sin \omega)^{2\lambda}} \int_{c_\omega} \frac{f(\theta', \varphi') ds'}{[\sin^2\theta' \sin^2 (\varphi - \varphi')]^{\frac{1}{2} - \lambda}},$$

and

$$(1.3) \qquad \varphi(\omega) = \left[f(\omega) - \frac{A\Gamma(\lambda)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2} + \lambda\right)} \right] (\sin \omega)^{2\lambda};$$

$$\Phi_p(x) = \frac{1}{\Gamma(p)} \int_0^x (x - t)^{p-1} \varphi(t) dt;$$

$$\Phi_0(x) = \varphi(x)$$

$$\varphi_p(x) = \Gamma(p + 1) x^{-p} \Phi_p(x), \qquad p \geq 0;$$

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and

$$\Phi_p(x) = \frac{d}{dx}\Phi_{p+1}(x), \qquad - < p < 0.$$

The absolute integrability of the integrand in (1.2) is assumed throughout.

The author has obtained theorems for CESÀRO summability of the series (1.1) analogous to those of IZUMI and SUNOUCHI [1]. The object of this paper is to extend and generalize the results of WANG [6 and 7] for the same series.

We prove the following:

THEOREM 1 - If $\nu > \beta > 0$ and

$$\Phi_\beta(t) = 0(t^{\nu+(1+\omega)2\lambda})$$

for $2\lambda\omega > 1$ and $0 < \lambda < 1$.

Then the series (1.1) is summable $(c, \alpha + \lambda)$ at the point $(0, \Phi)$ to the sum A, where

$$\alpha = \frac{\nu(m-1)+\beta}{\nu+m-\beta}$$

and m is a positive integer such that $m \geq \beta > m - 1$.

THEOREM 2. - If $\beta \geq 1$ and

$$\Phi_\beta(t) = 0\left(t^{\frac{\beta+2\lambda}{\alpha-\beta+1}}\right)$$

for $\beta - 1 < \alpha < \beta$ and $0 < \lambda < 1$

Then the series (1.1) is summable $(c, \alpha + \lambda)$ at the point (θ, Φ) to the sum A.

2. We require the following Lemmas:

LEMMA 1. - In order that the series (1.1) be summable (c, k) to the sum A, it is sufficient that the integral

$$i = \int_0^\delta \varphi(\omega) s_n^k(\omega) d\omega = 0(1)$$

for $0 < \delta < \pi$ and for each $k > \lambda$.

LEMMA 2. - Let $S_n^k(\omega)$ denote the n CESÀRO mean of order k of the series

$$\Sigma\,(n+\lambda)\,P_n^{(\lambda)}(\cos\omega).$$

Then we have, for $\lambda > 0$ and $p \geq 0$,

$$s_n^{(p)}(\omega) = \frac{d^p \{ s_n^k(\omega) \}}{d\omega^p} = \begin{cases} O(n^{2\lambda+p+1}) & \text{for } 0 \leq \omega \leq \pi, \ k > 0, \\ O\left(\dfrac{n^{\lambda+p-k}}{\omega^{k+\lambda+1}}\right) + O\left(\dfrac{1}{n\omega^{2\lambda+2+p}}\right) & \text{for } 0 < \omega \leq a < \pi; \\ O\left(\dfrac{n^{\lambda+p-k}}{\omega^{k+\lambda+1}}\right) & \text{for } 0 < \omega \leq a < \pi \text{ and } \lambda + 1 + [p] \geq k. \end{cases}$$

LEMMA 3. - For a non-integral

$$\delta = m + \sigma \qquad (0 < \sigma < 1),$$

we have

$$\int_0^{\Delta} \Phi_\delta(u) s_n^{(\delta)}(u) du = \Phi_{m+1}(\Delta) s_n^{(m)}(\Delta) \int_0^{\Delta} \Phi_m(t) s_n^{(m)}(t) dt.$$

LEMMA 4. - If $0 \leq u \leq \dfrac{1}{n}$

$$F(n, u) = O(n^{2\lambda+m+1} u^{m-\beta}) + O(n^{2\lambda+m+1} u^{m-\beta-1}).$$

where

$$F(n, u) = \Gamma \frac{1}{(m-\beta)} \int_u^{\frac{1}{n}} (t-u)^{m-\beta-1} s_n^{(m)}(t) dt.$$

Lemmas 1, 2, 3, are due to OBRECHKOFF [3] and Lemma 4 is known [5].

3. Proof of Theorem 1.

In order to prove the theorem, it is sufficient to show that

$$\lim_{n \to \infty} \int_0^{\delta} \varphi(u) s_n^{\alpha+\lambda}(u) du = 0$$

under the conditions of the theorem.

We have the following inequality:

$$\beta > \alpha > m-1 \qquad \text{[F. T. WANG]}$$

Also we have

$$\beta \leq m < \beta + 1.$$

Then

$$i = \int_0^{\delta} \varphi(\omega) s_n^k(\omega) d\omega$$

$$= \left[\sum_{p=1}^{m} (-1)^{p-1} \Phi_p(\omega) \left(\frac{d}{d\omega}\right)^{p-1} s_n^{\alpha+\lambda}(\omega) \right]_0^{\delta} + (-1)^m \int_0^{\delta} \Phi_m(t) s_n^{(m)}(t) dt.$$

$$= I + (-1)^m J.$$

since $\alpha > m-1$,

(3.1) $$I = o(1) \text{ as } n \to \infty.$$

We write

$$J = \int_0^{\delta} \Phi_m(t) s_n^{(m)}(t)\, dt$$

$$= \int_0^{\frac{1}{n}} + \int_{\frac{1}{n}}^{\delta} = J_1 + J_2.$$

(3.2) $$J_2 = \left[\Phi_m(t) s_n^{(m-1)}(t) \right]_{\frac{1}{n}}^{\delta} - \int_{\frac{1}{n}}^{\delta} \Phi_{m-1}(t) s_n^{(m-1)}(t) dt$$

$$= R - S.$$

$$R = o(1) + o\left(\frac{1}{n^{\nu \top (1+\omega)2\lambda}}\right) \cdot O\left(\frac{n^{m-1-\alpha}}{n^{-\alpha-1-2\lambda}}\right)$$

$$= o(1) \text{ as } n \to \infty.$$

Also if we write

$$\Phi^*(t) = \int_0^t |\Phi_{m-1}(u)|\, du.$$

Then

$$(3.3) \qquad S = O\left[n^{m-1-\alpha} \int_{\frac{1}{n}}^{\delta} \frac{|\Phi_{m-1}(t)|}{t^{\alpha+2\lambda+1}}\, dt\right]$$

$$= O(n^{m-1-\alpha}) \left[\frac{\Phi^*(t)}{t^{\alpha+2\lambda+1}}\right]_{\frac{1}{n}}^{\delta} + O(n^{m-1-\alpha}) \int_{\frac{1}{n}}^{\delta} \frac{\Phi^* t)}{t^{\alpha+2\lambda+2}}\, dt$$

$$= O(n^{m-1-\alpha}) + O(n^{m-1-\alpha}) \cdot o\left(\frac{1}{n^{\nu+(1+\omega)2\lambda}}\right) n^{\alpha + 2\lambda+1}$$

$$+ O(n^{m-1-\alpha}) \int_{\frac{1}{n}}^{\delta} \frac{O(t^{\nu+(1+\omega)2\lambda})}{t^{\alpha+2\lambda+2}}\, dt.$$

$$= o(1) \text{ as } n \to \infty.$$

When β is not an integer

$$J_1 = \int_0^{\frac{1}{n}} \Phi_m(t) s_n^{(m)}(t)\, dt$$

$$= \int_0^{\frac{1}{n}} \Phi_\beta(u) F(n, u)\, du.$$

where

$$F(n, u) = \frac{1}{\Gamma(m-\beta)} \int_u^{\frac{1}{n}} (t-u)^{m-\beta-1} s_n^{(m)}(t) dt.$$

Now

$$(3.4) \qquad J_1 = o\left[\int_0^{\frac{1}{n}} u^{\nu+(1+\omega)2\lambda} O(n^{2\lambda+m+1} u^{m-\beta}) du\right]$$

$$+ o\left[\int_0^{\frac{1}{n}} u^{\nu+(1+\omega)2\lambda} O(n^{2\lambda+m+1} u^{m-\beta-1}) du\right]$$

$$= o(1) \text{ as } n \to \infty.$$

When $\beta = m$ is an integer

$$(3.5) \qquad J_1 = \int_0^{\frac{1}{n}} \Phi_\beta(t) s_n^{(\beta)}(t) dt$$

$$= o\left[\int_0^{\frac{1}{n}} u^{\nu+(1+\omega)2\lambda} O(n^{2\lambda+\beta+1}) du\right]$$

$$= o(1) \text{ as } n \to \infty.$$

Combining (3.1), (3.2), (3.3), (3.4) and (3.5) the result is proved.

4. Proof of Theorem 2.

If we put

$$\beta = 1 + \delta,$$

and suppose for a non-integral δ

$$\delta = m + \sigma \qquad (o < \sigma < 1),$$

Also let us first take the case when $m \geq 1$, then

$$i = \int_0^\Delta \Phi(u) s_n^{\alpha+\lambda}(u) du$$

$$= \left[\sum_{\rho=1}^{m} (-1)^{\rho-1} \Phi_\rho(u) \left(\frac{d}{du}\right)^{\rho-1} s_n^{\alpha+\lambda}(u) \right]_0^\Delta + (-1)^m \int_0^\Delta \Phi_m(t) s_n^{(m)}(t) dt$$

$$= I + (-1)^m J.$$

From Lemma 2 we observe that

$$s_n^{(q)}(\Delta) = o(1) \text{ as } n \to \infty \text{ for } \alpha > q.$$

since $\alpha > \delta$ and $\delta > m$, hence $\alpha > m$.

Thus

(4.1) $$I = o(1) \text{ as } n \to \infty.$$

Now

$$J = \int_0^\Delta \Phi_m(t) s_n^{(m)}(t) dt$$

$$= \Phi_{m+1}(\Delta) s_n^{(m)}(\Delta) - \int_0^\Delta \Phi_\delta(u) s_n^{(\delta)}(u), \qquad \text{[by Lemma 3]}$$

$$= J_2 - J_3.$$

But

(4.2) $$J_2 = o(1) \qquad \text{since } \alpha > m.$$

Now

$$J_3 = \int_0^\Delta \Phi_\delta(u) s_n^{(\delta)}(u) du = \int_0^{\frac{1}{n^\gamma}} + \int_{\frac{1}{n^\gamma}}^\Delta = I_1 + I_2$$

where

$$r = \frac{\alpha - \delta}{\delta + 2\lambda + 1} = \frac{\alpha - \delta}{\beta + 2\lambda}$$

But we have [1].

$$(4.3) \qquad \varphi_\beta(t) = \frac{1}{t}\int_0^t \left(1 - \frac{u}{t}\right)^{\beta - 1} \varphi(u)du = 0\left(t^{\frac{\beta+2\lambda}{\alpha-\beta+1}1-\beta}\right)$$

If we put

$$\beta = 1 + \delta$$

then (4.3) is equivalent to

$$(4.4) \qquad \varphi_{1+\delta}(t) = o\left(t^{\frac{\beta+2\lambda}{\alpha-\beta+1}-\beta}\right)$$

so $\Phi\delta(u)$ is integrable in the sense of Cauchy–Lebesgue. Thus by (4.4), we get:

$$(4.5) \qquad \int_0^t \varphi_\delta(u)du = o\left(t^{\frac{\beta+2\lambda}{\alpha-\beta+1}-\beta+1}\right) = o\left(t^{\frac{\beta+2\lambda}{\alpha-\beta+1}-\delta}\right)$$

And so

$$(4.6) \quad \Phi^*(t) = \int_0^t \Phi_\delta(u)du$$

$$= \frac{1}{\Gamma(1+\delta)}\int_0^t u^\delta \varphi_\delta(u)du$$

$$= o\left(t^{\frac{\beta+2\lambda}{\alpha-\beta+1}}\right) \qquad \text{[by (4.5)].}$$

$$I_2 = \int_{\frac{1}{n^r}}^{\Delta} \Phi_\delta(u)s_n^{(\delta)}(u)du$$

$$= \int_{\frac{1}{n^r}}^{\Delta} O(1)\left[O\left(\frac{n^{\delta-\alpha}}{u^{\alpha+2\lambda+1}}\right) + O\left(\frac{1}{nu^{2\lambda+2+\delta}}\right)\right]du.$$

$$= O(n^{\delta-\alpha}) + O\left(n^{\delta-\alpha+\frac{\alpha-\delta}{\beta+2\lambda}(\alpha+2\lambda)}\right) + O\left(\frac{1}{n}\right) + O\left(\frac{1}{n}\cdot n^{r(2\lambda+\beta)}\right)$$

$$= o(1) \text{ as } n \to \infty.$$

(4.7)
$$\mathrm{I}_1 = \int_0^{\frac{1}{n^r}} \Phi_\delta(u) s_n^{(\delta)}(u)du = \int_0^{\frac{1}{n}} + \int_{\frac{1}{n}}^{\frac{1}{n^r}} = v + w$$

$$v = \int_0^{\frac{1}{n}} \Phi_\delta(u) s_n^{(\delta)}(u)du$$

$$= \left[\Phi^*(u) s_n^{(\delta)}(u)\right]_0^{\frac{1}{n}} - \int_0^{\frac{1}{n}} \Phi^*(u) s_n^{(\delta+1)}(u)du$$

$$= k_1 - k_2$$

$$k_1 = o\left[u^{\frac{\beta+2\lambda}{\alpha-\beta+1}}\, O(n^{2\lambda+\delta+1})\right]_0^{\frac{1}{n}}$$

$$= o\,(1) \text{ as } n \to \infty.$$

(4.8)
$$k_2 = o\left[\int_0^{\frac{1}{n}} u^{\frac{\beta+2\lambda}{\alpha-\beta+1}} \cdot O(n^{2\lambda+\delta+2})du\right]$$

$$= o(n^{\delta+2\lambda+2}) \cdot O\left(\frac{1}{n^{\frac{\beta+2\lambda}{\alpha-\beta+1}+1}}\right)$$

$$= o(1) \text{ as } n \to \infty.$$

(4.9)
$$w = \int_{\frac{1}{n}}^{\frac{1}{n^r}} \Phi_\delta(u) s_n^{(\delta)}(u)du$$

$$= \left[\Phi^*(u) s_n^{(\delta)}(u)\right]_{\frac{1}{n}}^{\frac{1}{n^r}} - \int_{\frac{1}{n}}^{\frac{1}{n^r}} \Phi^*(u) s_n^{(\delta+1)}(u) du$$

$$= k'_1 - k'_2$$

$$k'_1 = o\left[u^{\frac{\beta+2\lambda}{\alpha-\beta+1}} \left\{ O\left(\frac{n^{\delta-\alpha}}{u^{\alpha+2\lambda+1}}\right) + O\left(\frac{1}{n u^{2\lambda+2+\delta}}\right)\right\}\right]_{\frac{1}{n}}^{\frac{1}{n^r}}$$

$$= o(n^{\delta-\alpha}) \left[\frac{1}{n^{r\left\{\frac{\beta+2\lambda}{\alpha-\beta+1}-\alpha-2\lambda-1\right\}}}\right] + o(n^{\delta-\alpha}) \frac{1}{n^{\frac{\beta+2\lambda}{\alpha-\beta-1}-\alpha-2\lambda-1}}$$

$$= o(1) \text{ as } n \to \infty.$$

(4.10) $$k'_2 = \int_{\frac{1}{n}}^{\frac{1}{n^r}} o\left(u^{\frac{\beta+2\lambda}{\alpha-\beta+1}}\right) \cdot \frac{n^{\delta-\alpha+1}}{u^{\alpha+2\lambda+1}} du$$

$$= o\,(n^{\delta-\alpha+1}) \cdot \left[u^{\frac{\beta+2\lambda}{\alpha-\beta+1}-\alpha-2\lambda}\right]_{\frac{1}{n}}^{\frac{1}{n^r}}$$

$$= o(n^{\delta-\alpha+1}) \left[\frac{1}{n^{r\left\{\frac{\beta+2\lambda}{\alpha-\beta+1}-\alpha-2\lambda\right\}}}\right]$$

$$+ o\left[\frac{n^{\delta-\alpha+1}}{n^{\frac{\beta+2\lambda}{\alpha-\beta+1}-\alpha-2\lambda}}\right]$$

$$= o(1) \text{ as } n \to \infty.$$

Combining (4.1), (4.2), (4.3), (4.4), (4.5), (4.6), (4.7), (4.8), (4.9), and (4-10) the result is proved.

When δ is an integer, say $\delta = m$.

(4.11) $$i = \left[\sum_{\rho=1}^{m} (-1)^{\rho-1} \Phi_\rho(u) \left(\frac{d}{du}\right)^{\rho-1} s_n^{\alpha+\lambda}(u)\right]_0^{\Delta} + (-1)^m \int_0^{\Delta} \Phi_\delta(u) s_n^{(\delta)}(u) du$$

$$= o(1).$$

When $m = 0$

$$(4.12) \qquad \int_0^\Delta \Phi_0(u) s_n^{(0)}(u) du = \Phi_1(\Delta) s_n^{(0)}(\Delta) - \int_0^\Delta \Phi\delta(u) s_n^{(\delta)}(u) du$$

$$= o(1) \text{ as before.}$$

When $\beta = 1$ i.e. $\delta = 0$

$$i = \int_0^\Delta \Phi_0(u) s_n^{(0)}(u) du$$

and

$$\int_0^t \Phi_0(t) dt = o\left(t^{\frac{1+2\lambda}{\alpha}}\right)$$

By the help of (4.1), ...(4.10), we have

$$(4.13) \qquad i = o(1)$$

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