Bollettino Unione Matematica Italiana

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Bollettino dell'Unione Matematica Italiana, Serie 3, Vol. **16** (1961), n.2, p. 164–166.

Zanichelli

<http://www.bdim.eu/item?id=BUMI_1961_3_16_2_164_0>

On variation-diminishing properties of Green's functions.

Nota di RICHARD BELLMAN (a Santa Monica, USA) (*)

Summary. *The study of the properties of solutions of the boundary value problem*

$$u'' + q(x)u = f(x), \qquad u(0) = u(1) = 0,$$

as the forcing term $f(x)$ ranges over all elements in a function class is equivalent to the study of properties of the associated GREEN'*s function $K(x, y)$, This function enables us to write the solution in the form*

$$u(x) = \int_0^1 K(x, y) f(y) dy.$$

The question of determining the nonnegativity properties of $K(x, y)$, and more generally, the variation-diminishing properties, has been studied by a number of authors: KELLOGG, GANTMAKHER *and* KREIN; SCHOENBERG; ARONSZAJN *and* SMITH; KARLIN *and* MACGREGOR, *and others. In a brief note, it was shown that the nonnegativity of $K(x, y)$ could be deduced very simply from a consideration of the related quadratic form*

$$J(u) = \int_0^1 [u'^2 - q(x)u^2 + 2f(x)u] dx.$$

In this paper, let us show that the variation-diminishing property of the GREEN'*s function follows in the same simple fashion from the problem of the determination of the absolute minimum of $J(u)$.*

1. Introduction.

The study of the properties of solutions of the boundary value problem

$$(1) \qquad u'' + q(x)u = f(x), \qquad u(0) = u(1) = 0,$$

(*) Pervenuta alla Redazione dell' U. M. I. il 10 aprile 1961.

as the forcing term $f(x)$ ranges over all elements in a function class is equivalent to the study of properties of the associated GREEN's function $K(x, y)$. This function enables us to write the solution of (1) in the form

(2) $$u(x) = \int_0^1 K(x. \ y) f(y) dy.$$

The question of determining the nonnegativity properties of $K(x, y)$, and more generally, the variation–diminishing properties, has been studied by a number of authors : KELLOGG, GANTMAKHER and KREIN [1], SCHOENBERG [2], ARONSZAJN and SMITH [3], KARLIN and MACGREGOR [4], and others. In a brief note [5] it was shown that the nonnegativity of $K(x, y)$ could be deduced very simply from a consideration of the related quadratic form

(2) $$J(u) = \int_0^1 [u'^2 - q(x)u^2 + 2f(x)u] dx.$$

In this paper, let us show that the variation–diminishing property of the GREEN's function follows in the same simple fashion from the problem of the determination of the absolute minimum of $J(u)$.

2. Variation-diminishing Property.

The result we wish to establish is

THEOREM. - *Let* q(x) *satisfy the condition*

(1) $$q(x) \leq \pi^2 - d, \qquad d > 0,$$

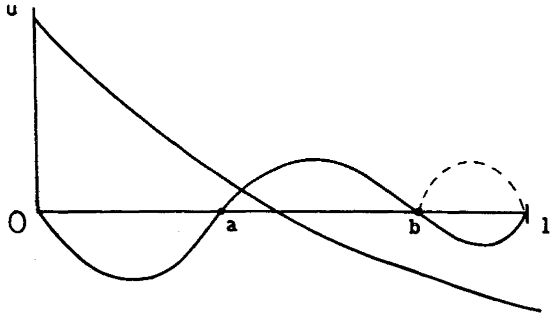
where π^2 *appears as the smallest characteristic value of the Sturm–Liouville problem*

(2) $$u'' + \lambda u = 0, \qquad u(0) = u(1) = 0.$$

Then, if q(x) *is as a continuous function with* N *changes of sign, the solution of* (1) *has at most* N *changes of sign.*

3. Proof of Theorem.

The EULER equation obtained from the first variation of $J(u)$ over all functions which vanish at $x = 0$ and $x = 1$ is (1.1). For the sake of simplicity, take the simplest case where $f(x)$ has one change of sign, and the solution $u(x)$ is assumed to have two changes of sign, as indicated below.



Were this situation to exist, we could obtain a smaller value for $J(u)$ by replacing $u(x)$ by a function which has the same values in the interval $[0, b]$, and is equal to the negative of $u(x)$ in $[b, 1]$. This change leaves the quadratic terms unchanged, and decreases the contribution of the tetm $2f(x)u$. This contradicts the fact that the solution of the EULER equation in this case yields the absolute minimum.

It is easy to see that the same argument handles the general cese where any number of changes of sign is allowed. We shall discuss the multidimensional case elsewhere.

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