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On the (H, p) summability of Fourier Series.

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Summary. - *Siddiqui, Hille and Tamarkin have applied harmonic summability method to Fourier Series. Here the author has defined a new process of summation, of Fourier Series, which generalises the harmonic summability introduced for the first time by Riesz.*

1. Let

$$\Phi(t) \sim \sum_{n=1}^{\infty} a_n \cos nt, \qquad \Phi(o) = 0. \tag{1.1}$$

be the FOURIER Series associated with an even function $\varphi(t)$ which is integrable in the sense of LEBESGUE over the interval $(0, 2\pi)$ and defined outside this range by periodicity with period 2π.

We write

$$\Phi(t) = \int_0^t |\varphi(u)| \, du. \tag{1.2}$$

DEFINITION. - If $s_n(t)$ be the n-th partial sum of the series (1.1), then we say that the FOURIER Series (1.1) is summable (H, p) to the sum zero at the point $t = 0$, if

$$\sigma_n(p, t) = \frac{\sum_{k=0}^{n} \left\{ s_{n-k}(t) \Big/ \prod_{q=0}^{p-1} (\log)^q (k+1) \right\}}{\{(\log)^p n\}} \tag{1.3}$$

tends to zero as $n \to \infty$, where p is a positive integer.

This generalises the Harmonic Summability of FOURIER Series, introduced by RIESZ [1]. HILLE and TAMARKIN [2] proved the following theorem of which a simple proof was later given by SIDDIQUI [3].

THEOREM H-T. - *If*

$$\Phi(t) = o\left(t/\log \frac{1}{t}\right) \tag{1.4}$$

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then the Fourier Series (1.1) *is summable by H rmonic means* – $(H,\ 1)$ *to the sum zero at the point* $t = 0$.

We shall prove the following:

THEOREM I. – *If*

$$\Phi(t) = 0\left(t \Big/ \prod_{q=0}^{p-1} \left\{ (\log)^{q+1} \frac{1}{t} \right\}\right) \tag{1.5}$$

then the Fourier Series (1.1) *is summable* (H, p) *to the sum zero at the point* $t = 0$.

Theorem I reduces to Theorem H-T when $p = 1$.

2. Transformation. – Since the n-th partial sum of the FOURIER Series (1.1) is given by

$$s_n(t) = \frac{1}{\pi}\int_0^\pi \varphi(t)\, \frac{\operatorname{Sin} nt}{t}\, dt. \tag{2.1}$$

Hence, $\sigma_n(p,\ t)$, the $(H,\ p)$ transform of (2.1), by (1.3), is

$$\sigma_n(p,\ t) = \int_0^\pi \frac{\varphi(t)}{t} \sum_{k=0}^{n} \frac{1}{\{(\log)^n\}}\, \frac{\operatorname{Sin}(n-k)t}{\left\{\prod_{q=0}^{p-1} (\log)^q (k+1)\right\}}\, dt \tag{2.2}$$

$$= \int_0^\pi \varphi(t) H_n(p,\ t)dt, \text{ say.}$$

where

$$H_n(p,\ t) = \frac{1}{\pi t \{(\log)^p n\}} \sum_{k=0}^{n} \frac{\operatorname{Sin}\ (n-k)t}{\left\{\prod_{q\ 0}^{p-1} (\log)^q (k+1)\right\}}. \tag{2.3}$$

3. We shall require the following lemmas.

LEMMA 1. – If $H_n(p,\ t)$ is defined as in (2.3), then for $0 \leq t \leq \frac{1}{n}$, we have

$$H_n(p,\ t) = O(n). \tag{3.1}$$

PROOF. - Since

$$H_n(p,\ t) = \frac{1}{\pi t \{\log)^p n\}} \sum_{k=0}^{n} \frac{\operatorname{Sin}(n-k)t}{\left\{\prod_{q=0}^{p-1} (\log)^q (k+1)\right\}} \tag{3.2}$$

$$= O\left(\frac{1}{t\,(\log)^p n}\right) \sum_{k=0}^{n} \frac{O(n-k)t}{\left\{\prod_{q=0}^{p-1} (\log)^q (k+1)\right\}}, \text{ since } 0 \leq t \leq \frac{1}{n}$$

$$= O(n) \frac{1}{\{(\log)^p n\}} \sum_{k=0}^{n} \frac{1}{\left\{\prod_{q=}^{p-1} (\log)^q (k+1)\right\}}$$

$$= O(n).$$

This proves the lemma, which generalises Lemma 3 due to SHARMA [4].

LEMMA 2. - For $\frac{1}{n} \leq t \leq \delta$,

$$H_n(p,\ t) = O\left(\frac{1}{t\,\{(\log)^p n\}}\right) \left\{1 + (\log)^p \frac{1}{t}\right\} \tag{3.3}$$

where δ is some positive constant.

PROOF. - We have from HARDY-ROGOSINSKY [5] and TITCHMARSH [6]

$$\left| \sum_{k=0}^{n} \frac{\operatorname{Sin}(k+1)t}{(k+1)} \right| = O(1), \tag{3.4}$$

and

$$\left| \sum_{k=0}^{n} \frac{\operatorname{Cos}(k+1)t}{(k+1)} \right| = O\left(\log \frac{1}{t}\right). \tag{3.5}$$

Therefore

$$H_n(p,\ t) = \frac{1}{\pi t\,\{(\log)^p n\}} \sum_{k=0}^{n} \frac{\operatorname{Sin}(n-k)t}{\left\{\prod_{q=0}^{p-1} (\log)^q (k+1)\right\}} \tag{3.6}$$

$$= \frac{\operatorname{Sin}(n+1)t}{\pi t\,\{(\log)^p n\}} \sum_{k=0}^{n} \frac{\operatorname{Cos}(k+1)t}{\left\{\prod_{q=0}^{p-1} (\log)^q (k+1)\right\}}$$

$$- \frac{\operatorname{Cos}(n+1)t}{\pi t\,\{(\log)^p n\}} \sum_{k=0}^{n} \frac{\operatorname{Sin}(k+1)t}{\left\{\prod_{q=0}^{p-1} (\log)^q (k+1)\right\}}$$

$$= O\left(\frac{\left\{(\log)^p \frac{1}{t}\right\}}{t \{(\log)^p n\}}\right) + O\left(\frac{1}{t\{(\log)^p n\}}\right)$$

$$= O\left(\frac{1}{t\{(\log)^p n\}}\right)\left(1 + (\log)^p \frac{1}{t}\right).$$

This proves the lemma.

LEMMA 3. - If $0 < \delta \leq t \leq \pi$

$$H_n(p, t) = O\left(\frac{1}{\{(\log)^p n\}}\right). \tag{3.7}$$

PROOF. - Since

$$H_n(p, t) = \frac{1}{\pi t \{(\log)^p n} \sum_{k=0}^{n} \frac{\operatorname{Sin}(n-k)t}{\left\{\prod_{q=0}^{p-1} (\log)^q (k+1)\right\}}. \tag{3.8}$$

By applying ABEL's transformation [ZYGMUND [7]] we get

$$H_n(p, t) = \frac{1}{\pi t \{(\log)^p n\}} \sum_{k=0}^{n} \left\{ \sum_{r=0}^{k} \operatorname{Sin}(n-r)t \right\} \left\{ \Delta \frac{1}{\prod_{q=0}^{p-1} \{(\log)^q (k+1)\}} \right\} \tag{3.9}$$

$$= \frac{1}{\pi t \{(\log)^p n\}} \sum_{k=0}^{n} \left\{ \frac{\operatorname{Cos} t/2 - \operatorname{Cos}\left(n - k + \frac{1}{2}\right)t}{\operatorname{Sin} t/2} \right\}$$

$$\left\{ \Delta \frac{1}{\left\{\prod_{q=0}^{p-1} (\log)^q (k+1)\right\}} \right\}$$

$$= O\left(\frac{1}{\{(\log)^p n\}}\right) \left\{ \sum_{k=1}^{n} O(1) \left[O\left(\frac{1}{k^2}\right) + o(1) \right] \right\}$$

$$= O\left(\frac{1}{\{(\log)^p n\}}\right).$$

This completes the proof of the lemma.

4. Proof of Theorem I. - By (2.2), we have

$$\sigma_n(p,\ t) = \int_0^{\pi} \varphi(t) H_n(p,\ t)dt \tag{4.1}$$

$$= \left(\int_0^{1/n} + \int_{1/n}^{\delta} + \int_{\delta}^{\pi}\right) \varphi(t) H_n(p,\ t)dt$$

$$= I_1 + I_2 + I_3, \text{ say.}$$

Now

$$I_1 = \int_0^{1/n} \varphi(t) H_n(p,\ t)dt \tag{4.2}$$

$$= \int_0^{1/n} |\ \varphi(t)\ | \cdot O(n)dt, \text{ by Lemma 1.}$$

$$= O(n)[\Phi(t)]_0^{1/n}$$

$$= O(n)\left[o\left(\frac{t}{\left\{ \prod_{q=0}^{p-1} (\log)^{q+1} \frac{1}{t} \right\}} \right) \right]_0^{1/n}, \text{ by (1.5)}$$

$$= o + o\left(\frac{1}{\left\{ \prod_{q=0}^{p-1} (\log)^{q+1} n \right\}} \right)$$

$$= o(1), \text{ as } n \to \infty.$$

Also

$$I_2 = \int_{1/n}^{\delta} \varphi(t) H_n(p,\ t)dt \tag{4.3}$$

$$= O(1) \int_{1/n}^{\delta} |\varphi(t)| \cdot O\left(\frac{1}{t\ \{(\log)^p n\}}\right)\left(1 + (\log)^p \frac{1}{t}\right), \text{ by Lemma 2}$$

$$= I_{2,1} + I_{2,2}, \text{ say.}$$

Considering $I_{2,1}$, we have

$$(4.4)\quad I_{2,1} = O\left(\frac{1}{(\log)^p n}\right)\int_{1/n}^{\delta}\frac{|\varphi(t)|}{t}\,dt$$

$$= O\left(\frac{1}{\{(\log)^p n\}}\right)\left\{\left[\frac{\Phi(t)}{t}\right]_{1/n}^{\delta} + \int_{1/n}^{\delta}\frac{\Phi(t)}{t^2}\,dt\right\}$$

$$= O\left(\frac{1}{\{(\log)^p n\}}\right)\left\{\left[o\left(\frac{1}{\left\{\prod_{q=0}^{p-1}(\log)^{q+1}\frac{1}{t}\right\}}\right)\right]_{1/n}^{\delta}\right.$$

$$\left. + \int_{1/n}^{\delta} o\left(\frac{1}{t\left\{\prod_{q=0}^{p-1}(\log)^{q+1}\frac{1}{t}\right\}}\right)dt\right\}, \text{ by (1.5)}$$

$$= o\left(\frac{1}{\{(\log)^p n\}}\right) + o\left(\frac{1}{\{(\log)^p n\}}\right)\left(\frac{1}{\left\{\prod_{q=0}^{p-1}(\log)^{q+1} n\right\}}\right)$$

$$+ o\left(\frac{1}{(\log)^p n}\right)\left[(\log)^{p+1}\frac{1}{t}\right]_{1/n}^{\delta}$$

$$= o(1), \text{ as } n \to \infty.$$

Next

$$(4.5)\quad I_{2,2} = O\left(\frac{1}{\{(\log)^p n\}}\right)\int_{1/n}^{\delta}\frac{|\varphi(t)|}{t}\left\{(\log)^p\frac{1}{t}\right\}dt$$

$$= O\left(\frac{1}{\{(\log)^p n\}}\right)\left[\frac{\Phi(t)}{t}\left\{(\log)^p\frac{1}{t}\right\}\right]_{1/n}^{\delta}$$

$$+ \int_{1/n}^{\delta}\Phi(t)\left[\frac{(\log)^p\frac{1}{t}}{t^2} + \frac{1}{t^2}\,\frac{1}{\left\{\prod_{q=0}^{p-2}(\log)^{q+1}\frac{1}{t}\right\}}\right]dt$$

$$= o\left(\frac{1}{(\log)^p n}\right) + o\left(\frac{1}{\left\{\prod_{q=0}^{p-1} (\log)^{q+1} n\right\}}\right)$$

$$+ o\left(\frac{1}{\{(\log)^p n\}}\right)\int_{1/n}^{\delta} \frac{\left\{(\log)^p \frac{1}{t}\right\} dt}{t\left\{\prod_{q=0}^{p-1} (\log)^{q+1}\frac{1}{t}\right\}}$$

$$+ o\left(\frac{1}{\{(\log)^p n\}}\right)\int_{1/n}^{\delta} \frac{1}{t\left\{\prod_{q=0}^{p-2} (\log)^{q+1}\left(\frac{1}{t}\right)\right\}}$$

$$\cdot \frac{dt}{\prod_{q=0}^{p-1}\left\{(\log)^{q+1}\left(\frac{1}{t}\right)\right\}}, \text{ by (1.5)}$$

$$= o\left(\frac{1}{\{(\log)^p n\}}\right) + o\left(\frac{1}{\left\{\prod_{q=0}^{p-1} (\log)^{q+1} n\right\}}\right)$$

$$+ o\left(\frac{1}{\{(\log)^p n\}}\right)\left\{\int_{1/n}^{\delta} \frac{dt}{t\left\{\prod_{q=0}^{p-2} (\log)^{q+1}\left(\frac{1}{t}\right)\right\}}\right.$$

$$\left. + \int_{1/n} \frac{dt}{t\left\{\prod_{q=0}^{p-1} (\log)^{q+1}\left(\frac{1}{t}\right)\right\}}\right\}$$

$$= o\left(\frac{1}{\{(\log)^p n\}}\right) + o\left(\frac{1}{\left\{\prod_{q=0}^{p-1} (\log)^{q+1} n\right\}}\right)$$

$$+ o\left(\frac{1}{\{(\log^p) n\}}\right)[O(1) + (\log)^p n]$$

$$+ o\left(\frac{1}{\{(\log)^p n\}}\right)\left[\left(\left\{(\log)^{p+1}\frac{1}{t}\right)\right\}\right]_{1/n}^{\delta}$$

$$= o(1), \text{ as } n \to \infty.$$

Thus from (4.3), (4.4) and (4.5). We have

$$(4.6) \qquad I_2 = o(1), \text{ as } n \to \infty.$$

Finally

$$I_3 = \int_\delta^\pi \varphi(t) H_n(p, t) dt \tag{4.7}$$

$$= \int_\delta^\pi | \varphi(t) | \cdot O \left\{ \frac{1}{(\log)^p n} \right\} \cdot dt \text{ by Lemma 3}$$

$$= O\left(| (\log)^p n | \right)$$

$$= o(1), \text{ as } n \to \infty.$$

By collecting (4.1), (4.2), (4.6) and (4.7), we get

$$\sigma_n(p, t) = o(1) \text{ as } n \to \infty. \tag{4.8}$$

This completes the proof of Theorem I.

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