BOLLETTINO
UNIONE MATEMATICA ITALIANA

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Bollettino dell'Unione Matematica Italiana, Serie 3, Vol. **16** (1961), n.2, p. 150–155.

Zanichelli

<http://www.bdim.eu/item?id=BUMI_1961_3_16_2_150_0>

Some generating functions for the Jacobi polynomials.

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Summary. - *Several generating functions closely related to the classical generating function for*

$$\sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x)t^n$$

are obtained; see in particular (7), (8), (9) *below. Making use of these formulas a number of relations involving the Jacobi polynomials are derived.*

1. Put

$$P_n^{(\alpha,\beta)}(x) = \binom{\alpha+n}{n} F\left(-n,\ n+\alpha+\beta+1\,;\ \alpha+1\,;\ \frac{1}{2}(1-x)\right). \tag{1}$$

The generating function

$$\sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x)t^n = 2^{\alpha+\beta}\rho^{-1}(1-t+\rho)^{-\alpha}(1-t+\rho)^{-\beta}, \tag{2}$$

$$\rho = (1-2xt+t^2)^{-\frac{1}{2}},$$

is familiar; for proofs of (1) see [3, p. 68] and [2, p. 127, problem 219].

In this note we shall obtain several additional generating functions closely related to (2). It is easily verified that

$$\frac{d}{dt}\{t^{\lambda}(1-t+\rho)^{-\lambda}(1+t+\rho)^{-\lambda}\} \tag{3}$$

$$= \lambda\rho^{-1}t^{\lambda-1}(1-t+\rho)^{-\lambda}(1+t+\rho)^{-\lambda},$$

$$\frac{d}{dt}\left(\frac{1-t+\rho}{1+t+\rho}\right)^{-\mu} = \frac{\mu}{\rho}\left(\frac{1-t+\rho}{1+t+\rho}\right)^{-\mu}. \tag{4}$$

(*) Pervenuta alla Segreteria dell'U. M. I. il 14 febbraio 1961.

We may put

$$(5)\qquad (1-t+\rho)^{-\alpha}(1+t+\rho)^{-\beta}=(1-t+\rho)^{-\lambda}(1+t+\rho)^{-\lambda}\cdot\left(\frac{1-t+\rho}{1+t+\rho}\right)^{-\mu},$$

where

$$(6)\qquad \lambda=\frac{1}{2}(\alpha+\beta),\qquad \mu=\frac{1}{2}(\alpha-\beta).$$

Using (3), (4), (5) we find that

$$2^{\alpha+\beta}\frac{d}{dt}\{t^{\lambda}(1-t+\rho)^{-\alpha}(1+t+\rho)^{-\beta}\}$$

$$=2^{\alpha+\beta}t^{\lambda-1}(\lambda+\mu t)\rho^{-1}(1-t+\rho)^{-\alpha}(1+t+\rho)^{-\beta}$$

$$=2^{\alpha+\beta}t^{\lambda-1}(\lambda+\mu t)\sum_{n=0}^{\infty}P_n^{(\alpha,\beta)}(x)t^n$$

$$=\sum_{n=0}^{\infty}\{P_n^{(\alpha,\beta)}(x)+\mu P_{n-1}^{(\alpha,\beta)}(x)\}t^{\lambda+n-1},$$

so that

$$(7)\qquad 2^{\alpha+\beta}(1-t+\rho)^{-\alpha}(1+t+\rho)^{-\beta}$$

$$=\sum_{n-0}^{\infty}\{\lambda P_n^{(\alpha,\beta)}(x)+\mu P_n^{(\alpha,\beta)}(x)\}\frac{t^n}{\lambda+n},$$

provided λ is not equal to 0 or a negative integer. When $\lambda\neq 0$ we have

$$(8)\qquad \left(\frac{1-t+\rho}{1+t+\rho}\right)^{-\alpha}=1+\alpha\sum_{n=1}^{\infty}P_{n-1}^{(\alpha,-\alpha)}(x)\frac{t^n}{n}.$$

Another special case of interest is

$$(9)\qquad 2^{2\alpha}(1-t+\rho)^{-\alpha}(1+t+\rho)^{-\alpha}=\sum_{n=0}^{\infty}\frac{\alpha}{\alpha+n}P_n^{(\alpha,\alpha)}(x)t^n,$$

provided α is not a negative integer.

We note also that (7) yields

$$(10) \qquad 2^{\alpha}(1-t+\rho)^{-\alpha} = \sum_{n=0}^{\infty} \frac{\alpha}{\alpha+2n} \{ P_n^{(\alpha,0)}(x) + P_{n-1}^{(\alpha,0)}(x) \} t^n.$$

$$(11) \qquad 2^{\beta}(1+t+\rho)^{-\beta} = \sum_{n=0}^{\infty} \frac{\beta}{\beta+2n} \{ P_n^{(0,\beta)}(x) - P_{n-1}^{(0,\beta)}(x) \} t^n.$$

In view of

$$P_n^{(\alpha,\beta)}(-x) = (-1)^n P_n^{(\beta,\alpha)}(x),$$

(10) and (11) are equivalent.

By means of (1) we can verify that

$$P_n^{(\alpha,0)}(x) + P_{n-1}^{(\alpha,0)}(x) = \frac{\alpha+2n}{\alpha+n} P_n^{(\alpha,-1)}(x),$$

so that (10) and (11) become

$$(12) \qquad 2^{\alpha}(1-t+\rho)^{-\alpha} = \sum_{n=0}^{\infty} \frac{\alpha}{\alpha+n} P_n^{(\alpha,-1)}(x) t^n$$

and

$$(13) \qquad 2^{\beta}(1+t+\rho)^{-\beta} = \sum_{n=0}^{\infty} \frac{\beta}{\beta+n} P_n^{(-1,\beta)}(x) t^n,$$

respectively.

2. It is not difficult to give a direct proof (that is, without assuming (2)) of such formulas as (8), (9) or (12). For example, to prove (9), we take

$$(1-t+\rho)(1+t+\rho) = 2(1-xt+\rho) = 2(1-xt) \left\{ 1 + \left(1 - \frac{(x^2-1)t^2}{(1-xt)^2} \right)^{\frac{1}{2}} \right\}.$$

Since [1, vol. 1, p. 101]

$$2^{\alpha} \{ 1 + (1-t)^{\frac{1}{2}} \}^{-\alpha} = \sum_{r=0}^{\infty} \frac{(\alpha)_{2r}}{r!\,(\alpha+1)_r} 2^{-2r} t^r,$$

we have

$$2^{2\alpha}(1-t+\rho)^{-\alpha}(1+t+\rho)^{-\alpha} = \sum_{r=0}^{\infty} \frac{(\alpha)_{2r}}{r\,!\,(\alpha+1)_r}\left(\frac{x^2-1}{4}\right)^r \frac{t^{2r}}{(1-xt)^{\alpha+2r}}$$

$$= \sum_{r=0}^{\infty} \frac{(\alpha)_{2r}}{r\,!\,(\alpha+1)_r}\left(\frac{x^2-1}{4}\right)^r t^{2r} \sum_{s=0}^{\infty} \frac{s\,!}{(\alpha+2r)_s} x^s t^s$$

$$= \sum_{n=0}^{\infty} (\alpha)_n t^n \sum_{2r\leq n} \frac{x^{n-2r}(x^2-1)^r}{2^{2r} r\,!\,(\alpha+1)_r (n-2r)\,!}.$$

Since [1, vol. 2. p. 175]

$$\sum_{2r\leq n} \frac{x^{n-2r}(x^2-1)^r}{2^{2r} r\,!\,(\alpha+1)_r (n-2r)\,!} = \frac{1}{(2\alpha+1)_n} C_n^{\left(\alpha+\frac{1}{2}\right)}(x)$$

$$= \frac{1}{(2\alpha+1)_n} \frac{(2\alpha+1)_n}{(\alpha+1)_n} P_n^{(\alpha,\alpha)}(x) = \frac{1}{(\alpha+1)_n} P_n^{(\alpha,\alpha)}(x),$$

(9) follows at once.

To prove (8) we take

$$\left(\frac{1+t+\rho}{1-t+\rho}\right)^{\alpha} = \left(1+\frac{2t}{1-t+\rho}\right)^{\alpha} = \sum_{r=0}^{\infty} (-1)^r \frac{(-\alpha)_r}{r\,!}\left(\frac{1-t+\rho}{2}\right)^{-r} t^r$$

$$= \sum_{r=0}^{\infty} (-1)^r \frac{(-\alpha)_r}{r\,!} \frac{t^r}{(1-t)^r} \sum_{s=0}^{\infty} \frac{(r)_{2s}}{s\,!\,(r+1)_s}\left(\frac{x-1}{2}\right)^s \frac{t^s}{(1-t)^{2s}}$$

$$\sum_{r,\,s=0}^{\infty} (-1)^r \frac{(-\alpha)_r (r)_{2s}}{(r+s)\,!\,s\,!}\left(\frac{x+1}{2}\right)^s t^{r+s} \sum_{j=0}^{\infty} \frac{(r+2s)_j}{j\,!} t^j$$

$$= \sum_{n=0}^{\infty} t^n \sum_{r+s+j=n} (-1)^r \frac{(-\alpha)_r (r)2s+j}{(r+s)\,!\,s\,!\,j\,!}\left(\frac{x-1}{2}\right)^s$$

$$= \sum_{n=0}^{\infty} t^n \sum_{s=0}^{n} \frac{1}{s\,!}\left(\frac{x-1}{2}\right)^s \sum_{r=0}^{n-s} (-1)^r \frac{(-\alpha)_r (r)_{n-r+s}}{(r+s)\,!\,(n-r-s)\,!}$$

$$= 1 + \alpha \sum_{n=1}^{\infty} t^n \sum_{s=0}^{n-1} \frac{1}{s!(s+1)!} \frac{(n+s-1)!}{(n-s-1)!} \left(\frac{x-1}{2}\right)^s$$

$$\cdot \sum_{r=0}^{n-1} \frac{(-\alpha+1)_r(-n+s+1)_r}{r!(s+2)_r}$$

$$= 1 + \alpha \sum_{n=1}^{\infty} t^n \frac{(\alpha+1)_{n-1}}{n!} \sum_{s=0}^{n-1} \frac{(-n+1)_s(n)_s}{s!(\alpha+1)_s} \left(\frac{x-1}{2}\right)^s$$

$$= 1 + \alpha \sum_{n=1}^{\infty} \frac{t^n}{n} P_{n-1}^{(\alpha, -\alpha)}(x).$$

3. Making use o(the formulas of ξ 1 numerous relations involving the Jacobi polynomial are easily obtained. For brevity we put

$$f_0^{(\alpha, \beta)}(x) = 1, \; f_n^{(\alpha, \beta)}(x) = \frac{\gamma P_n^{(\alpha, \beta)}(x) + \mu P_{n-1}^{(\alpha, \beta)}(x)}{\lambda + n} \; (n \geq 1), \tag{14}$$

where λ, μ are defined by (6). Then it follows immediately from (7) that

$$f_n^{(\alpha+\gamma, \beta+\delta)}(x) = \sum_{r=0}^{n} f_r^{(\alpha, \beta)}(x) f_{n-r}^{(\gamma, \delta)}(x). \tag{15}$$

Also using (2) and (7) we get

$$P_n^{(\alpha+\gamma, \beta+\delta)}(x) = \sum_{r=0}^{n} f_r^{(\alpha, \beta)}(x) P_{n-r}^{(\gamma, \delta)}(x). \tag{16}$$

Similarly it follows from (5), (7), (8), (9), that

$$f_n^{(\alpha, \beta)}(x) = \frac{\lambda}{\lambda+n} P_n^{(\lambda, \lambda)}(x) \tag{17}$$

$$+ \sum_{r=1}^{n-1} \frac{\lambda\mu}{r(\lambda+n-r)} P_{r-1}^{(\mu, -\mu)}(x) P_{n-r}^{(\lambda, \lambda)}(x).$$

As special cases of (15) we may mention

$$\frac{\alpha+\beta}{\alpha+\beta+n} P_n^{(\alpha+\beta,\alpha+\beta)}(x) = \sum_{r=0}^{n} \frac{\alpha\beta}{(\alpha+r)(\beta+n-r)} P_r^{(\alpha,\alpha)}(x) P_{n-r}^{(\beta,\beta)}(x), \tag{18}$$

$$\frac{\alpha+\beta}{n} P_{n-1}^{(\alpha+\beta,-\alpha-\beta)}(x) = \frac{\alpha}{n} P_{n-1}^{(\alpha,-\alpha)}(x) \tag{19}$$

$$+ \sum_{r=1}^{n-1} \frac{\alpha\beta}{r(n-r)} P_{r-1}^{(\alpha,-\alpha)}(x) P_{n-r-1}^{(\beta,-\beta)}(x) + \frac{\beta}{n} P_{n-1}^{(\beta,-\beta)}(x).$$

We have also

$$\sum_{r=0}^{n} f_r^{(\alpha,\beta)}(x) f_{n-r}^{(\beta,\alpha)}(x) = \frac{\alpha+\beta}{\alpha+\beta+n} P_n^{(\alpha+\beta,\alpha+\beta)}(x), \tag{20}$$

$$\sum_{r=0}^{n} f_n^{(\alpha,\beta)}(x) f_{n-r}^{(-\beta,-\alpha)}(x) = \frac{\alpha-\beta}{n} P_{n-1}^{(\alpha-\beta,\beta-\alpha)}(x) \quad (n \geq 1). \tag{21}$$

Also we note that

$$\sum_{r=0}^{n} f_r^{(\alpha,\beta)}(x) f_{n-r}^{(-\alpha,-\beta)}(x) = \begin{cases} 1 & (n=0) \\ 0 & (n<1), \end{cases} \tag{22}$$

$$\sum_{r=0}^{n} \frac{\alpha^2}{(\alpha+r)(\alpha+r-n)} P_r^{(\alpha,\alpha)}(x) P_{n-r}^{(-\alpha,-\alpha)}(x) = \begin{cases} 1 & (n=0) \\ 0 & (n<0). \end{cases} \tag{23}$$

A like formula involving $P_n^{(\alpha,-\alpha)}(x)$ is implied by (8).

Additional formulas can be obtained by using (12). For example we have from (7), (12), and (13).

$$f_n^{(\alpha,\beta)}(x) = \sum_{r=0}^{n} \frac{\alpha\beta}{(\alpha+r)(\beta+n-r)} P_r^{(\alpha,-1)}(x) P_{n-r}^{(-1,\beta)}(x). \tag{24}$$

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