Bollettino
Unione Matematica Italiana

Richard Bellman

On the iterative solution of two-point boundary value problems.

Bollettino dell'Unione Matematica Italiana, Serie 3, Vol. **16** (1961), n.2, p. 145–149.

Zanichelli

<http://www.bdim.eu/item?id=BUMI_1961_3_16_2_145_0>

Bollettino dell'Unione Matematica Italiana, Zanichelli, 1961.

On the iterative solution of two-point boundary value problems.

Nota di RICHARD BELLMAN (a Santa Monica, USA) (*)

Summary. - *The problem of solving* $x'' + A^2x = 0$, $x(0) = c$, $x(1) = d$, *can be solved in a straigtforward fashion involving the solution of a system of linear algebraic equations. It is shown that this can be avoided by the use of a simple iterative scheme involving only the solution of linear differential equations, and a minimum of storage of values.*

1. Introduction.

Consider the n-dimensional vector differential equation

$$x'' + A(t)x = 0 \tag{1}$$

where the solution is subject to the boundary conditions

$$x(0) = c, \; x(1) = d. \tag{2}$$

To obtain the solution in a straightforward way, we proced as follows. Let X_1 and X_2 be respectively the matrix solutions of

$$X'' + A(t)X = 0 \tag{3}$$

satisfying the initial conditions

$$\begin{aligned} X_1(0) &= 1, \; X_1'(0) = 0, \\ X_2(0) &= 0, \; X_2'(0) = 1. \end{aligned} \tag{4}$$

Letting g represent the unknown value of the derivative at $t = 0$,

(*) Pervenuta alla Segreteria dell'U. M. I. il 14 febbraio 1961.

$x'(0)$, we have

$$x = X_1(t)c + X_2(t)g. \tag{5}$$

From the relation

$$X_1(1)c + X_2(1)g = x(1) = d, \tag{6}$$

we obtain the value of g,

$$g = X_2(1)^{-1}[d - X_1(1)c] \tag{7}$$

provided that $X_2(1)$ is nonsingular. This is a characteristic value condition.

Turning to the computational aspects of the problem, we see that this mode of solution requires the solution of a set of n simultaneous linear algebraic equations. If n is large, say fifty or one hundred, even with today's digital computers certain questions of accuracy arise. The question arises as to whether or not there is a way of circumventing this procedure.

It is tempting to consider the following tehnique. Let us guess a value of $x'(0)$, say g, and solve ihe equation in (1) numerically using the *two* initial conditions, $x(0) = c$, $x'(0) = g$. Carrying through the integration, we obtain the values $x(1)$ and $x'(1)$. Let us retain the value of $x'(1)$ found in this way and use the correct boundary value $x(1) = d$ to integrate the equation numerically backwards to $t = 0$. In this way we obtain values of $x(0)$ and $x'(0)$. Retain the value of $x'(0)$ obtained in this way and use the correct value of $x(0)$, $x(0) = c$, to integrate the equation forwards to $t = 1$. We then iterate this procedure over and over.

It would be too much to expect that this method converges in all cases, and it is easy to show that it diverges in some cases. What is remarkable is that it works in any cases. The challenging problem is now to modify this technique in such a fashion as to make it yield the solution in the problem at hand.

We shall not discuss this important and difficult problem here.

2. Analysis of the Iterative Technique for Constant $A(t)$.

Let us consider the equation

$$x'' + A^2 x = 0, \tag{1}$$

subject to the conditions $x(0)=c$, $x(1)=d$, under the assumption that A^2 is a positive definite matrix.

Taking an arbitrary value for $x'(0)$, g, we can write the solution in the form

$$x = X_1(t)c + X_2(t)g, \tag{2}$$

where

$$X_1(t) = \cos At + (e^{iAt} + e^{-iAt})/2, \tag{3}$$

$$X_2(t) = A^{-1} \sin At = A^{-1}(e^{iAt} - e^{-iAt})/2i,$$

and e^{iAt} is the matrix exponential; cf. [1].

Then

$$x'(1) = X_1'(1)c + X_2'(1)g. \tag{4}$$

Starting the solution at $t=1$ with the values $x(1)=d$ and the preceding value of $x'(1)$, we have

$$x = dX_1(1-t) - (X_1'(1)c + X_2'(1)g)X_2(1-t), \tag{5}$$

since $X_1(1-t)$, $-X_2(1-t)$, are the fundamental solutions for $t=1$.

Then

$$x'(0) = -dX_1' + (X_1'(1)c + X_2'(1)g)X_2'(1). \tag{6}$$

Hence if g_n is the $n-th$ approximation to $x'(0)$, we have the recurrence relation

$$g_{n+1} = -dX_1'(1) + (X_1'(1)c + X_3'(1)g_n)X_2'(1). \tag{7}$$

If g_n converges to a quantity g, this quantity is given by

$$g = [1 - X_2'(1)^2]^{-1}[X_1'(1)X_2'(1)c - dX_1'(1)]. \tag{8}$$

On the other hand. the solution of the two-point boundary value problem, obtained as in §1. is given by

$$g = X_2(1)^{-1}[d - X_1(1)c]. \tag{9}$$

To see the identity of the two expressions, observe that

$$X_1'(1) = -A \sin A,\ X_2'(1) = \cos A = X_1(1). \tag{10}$$

Since $X_1(1)$, $X_2(1)$, $X_1'(1)$, $X_2'(1)$ all commute, the expression in (8) may be written

$$\begin{aligned} g &= (\sin A)^{-2}(-A \sin A)(X_1(1)c - d) \\ &= X_2(1)^{-1}(d - X_1(1)c). \end{aligned} \tag{11}$$

We see then that we obtain the solution of our problem, provided that the recurrence relation of (7) yields a convergent sequence. This will be the case if all the characteristic roots of $X_2'(1)$ are less than 1 in absolute value. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the characteristic roots of A, then $\cos\lambda_1, \cos\lambda_2, \ldots, \cos\lambda_n$ are the roots of $\cos A$. Hence if $\cos\lambda_i \neq 1$ for $i = 1, 2, \ldots, n$, a characteristic value condition required for the solution given by (9), the iteration procedure yields the solution.

A similar analysis applies to the case where the boundary conditions are

$$x(0) = c,\ x'(1) = d. \tag{12}$$

3. General Case.

It is clear that we cannot expect convergence in general. If we replace (2.1) by the equation

$$x'' - A^2 x = 0, \tag{1}$$

with $x(0) = c$, $x(1) = d$, we see that $X_2'(1) = \cosh A = (e^A + e^{-A})/2$. Hence *all* the characteristic roots of $X_2'(1)$ are greater than one, and the iteration process always diverges.

On the other hand, the solution of

$$x'' - A^2x = 0, \; x(0) = c, \; x'(t_0) = d, \tag{2}$$

can be found by this technique, provided that t_0 is sufficiently small, and the same holds for the equation

$$x'' + A(t)x = 0, \; x(0) = c, \; x'(t_0) = d. \tag{3}$$

This means that the computational solution of the problem of minimizing the functional

$$J(x) = \int_0^{t_0} [(x', x') - (x, A(t)x)]dt. \tag{4}$$

over all functions for which $x(0) = c$ can be obtained in the foregoing fashion, provided t_0 is sufficiently small.

This, in turn, enables us to conclude the general variational problem of minimizing

$$J(x) = \int_0^{t_0} g(x, x')dt, \tag{5}$$

over all functions with $x_0 = c$, can be treated in this simple fashion if t_0 is sufficiently small.

The challenging problem is that of developing similar iterative schemes involving only the solution of differential equations which will yield the solution whenever it exists.

REFERENCE

[1] R. BELLMAN, *Introduction to Matrix Analysis*, McGraw-Hill Book Company, Inc., New York, 1960, Chapter X.