Bollettino
Unione Matematica Italiana

T. A. A. Broadbent

The concept of inequality.

Bollettino dell'Unione Matematica Italiana, Serie 3, Vol. **16** (1961), n.1, p. 79–83.

Zanichelli

<http://www.bdim.eu/item?id=BUMI_1961_3_16_1_79_0>

L'utilizzo e la stampa di questo documento digitale è consentito liberamente per motivi di ricerca e studio. Non è consentito l'utilizzo dello stesso per motivi commerciali. Tutte le copie di questo documento devono riportare questo avvertimento.

Articolo digitalizzato nel quadro del programma bdim (Biblioteca Digitale Italiana di Matematica) SIMAI & UMI
http://www.bdim.eu/

Bollettino dell'Unione Matematica Italiana, Zanichelli, 1961.

The concept of inequality.

Nota di T. A. A. Broadbent (a Londra)

Summary. - *Ideas about « less » and « more » are as fundamental as ideas about precise equality, and should enter the teaching of mathematics at an early stage, leading to the notion of an equality as the boundary between two inequality domains.*

This note is offered in homage to Professor Enrico Bompiani, in the year of his scientific jubilee, to express admiration for his work in mathematics and for the teaching of mathematics.

In a short communication to the International Congress of Mathematicians, Edinburgh 1958, entitled « Equal and unequal », Dr. R. C. Tanner made a plea for an earlier recognition of the importance of inequality in the teaching of mathematics to the child. She remarked that the sign of equality, devised in the 16th century, preceded the familiar inequality signs, which did not appear till the 17th century; yet this reverses the historic order, since concepts of « more » and « less », that is, of inequality, precede the notion of precise equality. A very small child will soon learn to use the word « more » with purpose and accuracy, long before the idea of equality becomes known. Such words are more primitive than the notion of equality; equality itself is a negative and derivative concept, since we arrive at it from the idea of « neither more nor less ». Indeed, in mature mathematics, an absolute equality is often best proved by a *reductio ad absurdum*, that is, by discussing a related inequality. Equality thus indicates a boundary, while inequalities are concerned with what lies on one side or the other of that boundary. Dr. Tanner maintained that we should emphasise these ideas in the early years of our pupils, in exercises and games which ask what is bigger or smaller. This emphasis should be continued in the later school years, where the teacher should make full use of what the child already knows, almost intuitively, about inequality.

It may be argued that inequality is not substantially a much more primitive notion than equality, since as soon as primitive man was concerned with heaps too large to be compared by eye, he must have been forced to compare them by a one-to-one matching, and then by a one-to-one matching with a standard heap, such as is provided by the fingers and toes, and this is effectively using a concept of equality. But this does not significantly affect the main contention, which is not that there should be a marked priority for inequalities, but merely that fair emphasis should be given to them from the earliest stages of teaching onwards. If the child can be encouraged to develop his intuitive ideas of «more» and «less», then in the later scohol years the present inept handling and fear of inequalities can be eradicated.

The notion of an equality as a *boundary*, a boundary which may be important largely because of the domains which lie on each side of it, has great value in many elementary fields. For instance, the comprehension of the simple geometry of the plane can be deepened. The equation, the equality $ax + by + c = 0$, determines a straight lin in the plane; this is familiar enough, though in England perhaps not enough stress is placed from the earliest stages on the linearity of the equation. Much deeper understanding of the topic comes to those who think rather of the *linear form* $ax + by + c$ with its associated line which divides the plane into two parts, in one of which $ax + by + c > 0$, in the other of which $ax + by + c < 0$, the equality sign picking out those points which form the boundary of the two regions. We may prove this in the early stages by some informal geometrical argument, and, better. as soon as possible as an immediate corollary to the JOACHIMSTHAL ratio equation when applied to a straight line or linear form, but however proved it will greatly enrich the content of the subject at a quite elementary level, and incidentally do something towards improving the clumsy or careless behaviour of the average pupil when confronted with an ambiguity of sign. In the same way, we must recognise the circle $x^2 + y^2 = a^2$ at an early stage as a boundary, dividing those interior points for which $x^2 + y^2 < a^2$ from the exterior points for which $x^2 + y^2 > a^2$. Such apparently trivial matters concerning the division of the plane have their applications at later stages: in grasping the shape and character of a curve, a matter which occurs quite often in naval architecture; in the elements of conformal mapping;and in dealing with the location and nature of complex roots of an equation.

In a quite different connection, emphasis on the concept of inequality in the early school years may help to ease the difficulties which many pupils encounter in dealing with elementary friction problems in statics. A pupil is asked to discuss a statical problem in which equilibrium can be broken in one way if a certain inequality is satisfied and in another way if the inequality is reversed, or a problem seeking to determine the condition under which equilibrium will be maintained, in the form of an inequality governing the magnitude of the coefficient of friction. Few teachers or examiners will deny that the average pupil will begin by assuming a state of limiting equilibrium and thence obtain an equality, from which he will hope to determine the sense of the associated inequalities by arguments certainly intuitive and often dubious. He will work *from* the limit rather than *to* the limit, he will think of the boundary rather than of the regions determined by the boundary; so much so, that cynical examiners have asserted that if by a printer's error the required inequalities appeared reversed, the same arguments would be confidently advanced to establish the incorrect results. Much of this confusion of thought can be avoided by empasising that friction is essentially concerned with an inequality, not an equality, that the vital principle, in the simplest case, is that if F is the friction and R the normal reaction, then F can not exceed μR, with equality only for the special limiting or boundary case.

In dealing with the number field, a child at school will often display some instinctive repugnance to the notion of the irrational number, since it does not appear to him to have the same sharp, concrete reality as has the integer or the rational. Now whether the irrational is defined by the method of DEDEKIND, of CANTOR, or of WEIERSTRASS, the concept of inequality and indeed of an infinite set of inequalities is present, explicitly; we can in fact find the origin of such sets as far back as the FIFTH BOOK of EUCLID, with its definitions of «equal ratio» and «greater ratio». No one, even in England, would advocate a return to EUCLID, but genuine familiarity with the idea of an infinite set of inequalities would clarify and strengthen the child's comprehension of the irrational number. Here indeed the DEDEKIND definition is related immediately to the concept of equality as a boundary between two regions of inequality, but all three methods imply the existence of infinitely many inequalities, and a recognition of the indirect, negative nature of precise and absolute equality. Such recognition has a practical value; it is not unknown, for example, for the practical man to ask the mathematician for the positive root of an equation, say,

$$x^3 - 2x - 5 = 0,$$

and when told that the root is between 2.09455 and 2.09456, to reply that this is not the answer he wants, that what he wants is the *exact* answer. That such a reply can be made implies a teaching defect somewhere, which might have been avoided by emphasis at the proper time on the value, theoretical and practical, of an infinite set of inequalities. Such emphasis will ensure that to a question about the value of e. the reply that it satisfies all inequalities

$$\left(1+\frac{1}{n}\right)^{n} < e < \left(1-\frac{1}{n}\right)^{-n} \qquad (n \text{ a positive integer})$$

will be recognised as an answer and not rejected as it often now is, as an evasion.

In courses of mathematics for technical students, of the junior grade, the concepts and techniques of elementary inequalities are often ignored, possibly on the grounds that they belong to pure mathematics and have little technological application; this is a mistake. In the numerical answer which a technical problem usually requires, all that can usually be given is an approximation, and bounds to such an approximation must generally be supplied before the answer can have any genuine significance. In other words, we should think, not as we so often do of a particular and identifiable point on a scale, but of sections of the scale, domains of significant approximation, regions determined by bounding inequalities. Viewed in this way, the pupil learns to see that adjunction of maximum possible error to an approximation is not a tiresome requirement insisted on by the pedantry of the mathematician, but is an essential item in a numerical answer; such adjunction will also serve to emphasise the fact that in practical problems, the numerical data can as a rule only be properly specified by inequalities, not by equalities.

At some stage of the school course, the formal study of the two basic rules of manipulation must be given:

1° For real $a, b, c,$ $\quad a > b$ implies $a + c > b + c$;

2° (*i*) If $c > 0,$ $\quad a > b$ implies $ac > bc$;

(*ii*) If $c < 0,$ $\quad a < b$ implies $ac < bc$.

If we think in terms of inequalities defining regions on a straight line, so that a is in the region to the right of b, then 1° is obvious, since a simple translation c to right or left does not affect relative position. In 2° (*i*), we have a simple magnification, and again relative position is not affected; in 2° (*ii*), there is a magnification together with a reversal (or rotation through 180°), and relative positions are simply reversed.

A very few fundamental inequalities are needed in the school. It has long been known, but is still often ignored in elementary teaching, that these, including for instance the CAUCHY–HÖLDER inequality if desired, can all be obtained from JENSEN'S theorem: if positive loads w are placed at points (x_r, y_r) on a curve which is concave upwards, then the centroid of these loads lies above the curve. As every mathematician knows, this can be made to provide all the inequalities ever needed by the schoolboy and most of those required by the undergraduate. Its statement and its proof, at any rate in a form adequate for school purposes, are both well within the comprehension of the good schoolboy; it could well be more widely used in schools than it appears to be.

In thanking DR. TANNER for allowing me to add these comments to her text, I must add the hope that some experienced teacher at the primary level will experiment with the notion of encouraging young children to pay greater attention to ideas of « more » and « less » and to make these as fundamental as the idea of equality.