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A note on the Borel summability of Fourier series

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Snmmary. - *Sinvhal has considered the Borel Summability of the Conjugate Series of the Derived Series, assuming two conditions on the function i. e. firstly - bounded variation and secondly-continnity one. The author in this note has determined the Borel Summability of the Fourier Series assuming only continuity condition.*

1. Let $f(t)$ be a function integrable in the sense of LEBESGUE over the interval $(0, 2\pi)$ and periodic with period 2π outside this interval. Let the series

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \operatorname{Sin} nx) \tag{1.1}$$

be the FOURIER Series associated with function $f(t)$, we further write

$$\Phi(t) = \frac{1}{2}\{f(x+t) + f(x-t) - 2s\}. \tag{1.2}$$

BOREL Summability of FOURIER Series was considered by STONE [1]. SINVHAL [2] considered the BOREL Summability of the conjugate Series of the Derived Series of (1.1). KNOPP [3] has defined the generalised BOREL Summability. The object here is to prove the following theorem.

THEOREM. *If*

$$\int_0^t |\Phi(u)|\, du = 0\left(t/\log\frac{1}{t}\right) \tag{1.3}$$

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then the Fourier Series (1.1) *is summable by Borel means* (*also called summable-B) to the sum zero at the point* t = x.

2. We shall prove the Theorem as follows:

PROOF. - Since, the, n^{th} partial sum of the series (1.1) is given by, see GERGEN [4],

$$s_n(t) = \frac{1}{\pi}\int_0^{\pi} \Phi(t) \frac{\operatorname{Sin} nt}{t} dt. \tag{2.1}$$

Now following HARDY [5], the BOREL transform of $s_n(t)$ is given by

$$\sigma_p(t) = \frac{e^{-p}}{\pi}\int_0^{\pi} \frac{\Phi(t)}{t}\left(\sum_{n=0}^{\infty} \frac{p^n \operatorname{Sin} nt}{Ln}\right) dt$$

$$= \frac{1}{\pi}\int_0^{\pi} \frac{\Phi(t)}{t} \cdot \frac{\operatorname{Sin}(p \operatorname{Sin} t)}{\exp\{p(1-\cos t)\}} dt$$

$$= \frac{1}{\pi}\left(\int_0^{1/p} + \int_{1/p}^{yp^{\alpha}} + \int_{1/p^{\alpha}}^{\pi}\right)\frac{\Phi(t)}{t}\, \frac{\operatorname{Sin}(p \operatorname{Sin} t)}{\exp\{p(1-\cos t)\}} dt, \text{ where } 0 < \alpha < \frac{1}{2}.$$

$$= I_1 + I_2 + I_3, \text{ say.} \tag{2.2}$$

Now

$$I_1 = \frac{1}{\pi}\int_0^{1/p} \frac{\Phi(t)}{t} \cdot \frac{\operatorname{Sin}(p \operatorname{Sin} t)}{\exp\{p(1-\cos t)\}} \cdot dt$$

$$= O(1)\int_0^{1/p} \frac{|\Phi(t)|}{t} O(pt) dt$$

$$= O(p)\int_0^{1/p} |\Phi(t)|\, dt$$

$$= O(p)\left[0\left(t/\log\frac{1}{t}\right)\right]_0^{1/p}, \quad \text{by (1.3).}$$

$$= 0 + 0\left(\frac{1}{\log p}\right)$$

$$(2.3) \qquad = 0(1), \quad \text{as } p \to \infty.$$

Next, we have

$$I_2 = \frac{1}{\pi}\int_{1/p}^{1/p^{\alpha}} \frac{\Phi(t)}{t} \frac{\operatorname{Sin}(p \operatorname{Sin} t)}{\exp\{p(1-\cos t)\}} \cdot dt$$

$$= \frac{1}{\pi \exp\left\{p \cdot 2 \operatorname{Sin}^2 \frac{p}{2}\right\}} \int_{1/p}^{1/p^{\alpha'}} \frac{\Phi(t)}{t} \cdot \operatorname{Sin}(p \operatorname{Sin} t) dt, \text{ for } 0 < \alpha < \alpha' < \frac{1}{2},$$

by second Mean value Theorem.

$$= O(1)\int_{1/p}^{1/p^{\alpha'}} \frac{|\Phi(t)|}{t} \cdot dt$$

$$= O(1)\left[\frac{\Phi(t)}{t}\right]_{1/p}^{1/p^{\alpha'}} + O(1)\int_{1/p}^{1/p^{\alpha'}} \frac{\Phi(t)}{t^2} \cdot dt$$

$$= O(1)\left[0\left(\frac{1}{\log \frac{1}{t}}\right)\right]_{1/p}^{1/p^{\alpha'}} + O(1)\int_{1/p}^{1/p^{\alpha'}} 0\left(\frac{dt}{t \log \frac{1}{t}}\right), \quad \text{by (1.3)}$$

$$= 0\left(\frac{1}{\log p}\right) + 0\left(\log \log \frac{1}{t}\right)_{1/p}^{1/p^{\alpha'}}$$

$$= 0(1/\log p) + 0(\log \alpha')$$

$$(2.4) \qquad = 0(1), \quad \text{as } p \to \infty.$$

Lastly, we have

$$I_3 = \frac{1}{\pi}\int_{1/p^{\alpha}}^{\pi} \frac{\Phi(t)}{t} \cdot \frac{\text{Sin}\,(p\,\text{Sin}\,t)}{\exp\{p(1-\cos t)\}}\,dt$$

$$= \frac{p^{\alpha}}{\pi \exp\left\{p \cdot 2\,\text{Sin}^2 \frac{1}{p^{\alpha}}\right\}} \int_{1/p^{\alpha}}^{\delta} \Phi(t)\,\text{Sin}\,(p\,\text{Sin}\,t)\,dt, \text{ where } \frac{1}{p^{1/2}} < \delta < \pi,$$

by second mean value Theorem.

$$= O(1)\left[(p^{\alpha}) / \exp\{p^{1-2\alpha}\}\right] \int_{1/p^{\alpha}}^{\delta} |\,\Phi(t)\,|\,dt$$

$$= O(1)\left[\frac{p^{\alpha}}{\exp\{p^{1-2\alpha}\}}\right] \cdot O(1),$$

by the continuity part of the integral $\int |\,\Phi(t)\,|\,dt$.

(2.5) $$= 0(1), \quad \text{as } p \to \infty, \quad \text{since } 0 < \alpha < \frac{1}{2}.$$

Thus, on collection (2.2), (2.3), (2.4) and (2.5), we find that

(2.6) $$\sigma_p(t) = 0(1), \quad \text{as } p \to \infty.$$

This completes the proof.

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