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Note on bilinear generating functions for the Laguerre polynomials.

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Summary. - *Polynomial identities, equivalent to certain bilinear generating functions for the Laguerre polynomials, are obtained.*

1. It is not difficult to verify that the formula

$$\sum_{n=0}^{\infty} \frac{n!}{(\alpha+1)_n} L_n^{(\alpha)}(x) L_n^{(\alpha)}(y) t^n \tag{1}$$

$$= \Gamma(\alpha+1)(1-t)^{-1} \exp\left(-\frac{(x+y)t}{1-w}\right)(xyw)^{-\alpha/2} I_\alpha\left(\frac{2(xyw)^{\frac{1}{2}}}{l-w}\right)$$

is equivalent to

$$L_n^{(\alpha)}(x) L_n^{(\alpha)}(y) = \frac{(\alpha+1)_n}{n!} \sum_{r=0}^{n} (xy)^r \frac{L_{n-r}^{(\alpha+2r)}(x+y)}{r!\,(\alpha+1)_r} \tag{2}$$

(compare Bailey [1, p. 219]. The writer [4] showed that the following formula of Bateman [2, p. 457], [3, p. 144]

$$\sum_{k=0}^{\infty} \frac{n!}{k!} (xye^{i\Phi})^{k-n} L_n^{(k-n)}(x^2) L_n^{(k-n)}(y^2) \tag{3}$$

$$= \exp(xye^{i\Phi}) L_n(x^2 + y^2 - 2xy\cos\Phi),$$

where

$$L_n^{(\alpha)}(x) = \sum_{r=0}^{n} \binom{n+\alpha}{r} \frac{(-x)^{n-r}}{(n-r)!} \quad (L_n(x) = L_n^{(0)}(x)) \tag{4}$$

is the Laguerre polynomial of degree n, is equivalent to the

(*) Pervenuta alla Segreteria dell'U. M. I. il 20 gennaio 1961.

polynomial identity

$$(5)\qquad (xy)^n L_n(x^2+y^2-2xy\cos\Phi)$$

$$= n!\cdot\sum_{k=0}^{n-1}(xy)^k\frac{\Delta_k}{k!}\,2\cos(n-k)\Phi+(xy)^n\Delta_n,$$

where

$$\Delta_k=\sum_{r=0}^{k}(-1)^{k-r}\binom{k}{r}L_n^{(r-n)}(x^2)L_n^{(r-n)}(y^2),$$

Moreover (5) was generalized to

$$(6)\qquad (\lambda)_{n+1}L_n^{(\lambda)}(x^2+y^2-2xyz)$$

$$=\sum_{r=0}^{n}(xy)^{n+r}(n+\lambda-r)\frac{\Delta_r^{(\lambda+n)}}{r!}C_{n-r}^{(\lambda)}(z)$$

where λ is arbitrary.

$$(7)\qquad \Delta_r^{(\lambda)}=\sum_{s=0}^{r}(-1)^r\binom{r}{s}(s!)^2(xy)^{-2s}L_s^{(\lambda-s)}(x^2)L_s^{(\lambda-s)}(y^2)$$

and

$$(1-2zt+t^2)^{-\lambda}=\sum_{n=0}^{\infty}C_n^{(\lambda)}(z)t^n.$$

(Note that the factor $(xy)^{-2s}$ in (7) was omitted in (3.4) of [4]).

2. The formula [3, p. 151]

$$(8)\qquad \sum_{n=0}^{\infty}n!\,L_n^{(\alpha-n)}(x)L_n^{(\beta-n)}(y)t^n$$

$$=\beta!\,e^{xyt}(1-yt)^{\alpha-\beta}t^\beta L^{(\alpha-\beta)}\left(-\frac{(1-xt)(1-yt)}{t}\right)$$

holds for all x, y, t provided α, β are non-negative integers; the writer [5] has recently given a simple proof of (8). It may be of interest to obtain a polynomial identity equivalent to (8).

Using (4) we have

$$\beta\,!\,(1-yt)^{\alpha-\beta}t^{\beta}L_{\beta}^{(\alpha-\beta)}\left(-\frac{(1-xt)(1-yt)}{t}\right)$$

$$=\beta\,!\,(1-yt)^{\alpha-\beta}t^{\beta}\sum_{r=0}^{\beta}\binom{\alpha}{r}\frac{1}{(\beta-r)\,!}\left(\frac{(1-xt)(1-yt)}{t}\right)^{\beta-r}$$

$$=\sum_{r=0}^{\min(\alpha,\beta)}r\,!\binom{\alpha}{r}\binom{\beta}{r}t^{r}(1-xt)^{\beta-r}(1-yt)^{\alpha-r}$$

$$=\sum_{r=0}^{\min(\alpha,\beta)}r\,!\binom{\alpha}{r}\binom{\beta}{r}t^{r}\sum_{j=0}^{\beta-r}(-1)^{j}\binom{\beta-r}{j}(xt)^{j}$$

$$\cdot\sum_{k=0}^{\alpha-r}(-1)^{k}\binom{\alpha-r}{k}(yt)^{k}$$

$$=\sum_{n=0}^{\infty}t^{n}\sum_{j+k+r=n}(-1)^{j+k}r\,!\binom{\alpha}{r}\binom{\beta}{r}\binom{\alpha-r}{k}\binom{\beta-r}{j}x^{j}y^{k}$$

$$=\sum_{n=0}^{\infty}t^{n}\sum_{j+k\le n}(-1)^{j+k}\binom{\alpha}{r+k}\binom{\beta}{r+j}\frac{(r+k)\,!\,(r+j)\,!}{r\,!\,k\,!\,j\,!}x^{j}y^{k}$$

$$=\sum_{n=0}^{\infty}t^{n}\sum_{j+k\le n}(-1)^{j+k}\binom{\alpha}{n-j}\binom{\beta}{n-k}\frac{(n-j)\,!\,(n-k)\,!}{j\,!\,k\,!\,(n-j-k)\,!}x^{j}y^{k}.$$

Thus (8) is equivalent to

$$\sum_{r=0}^{n}(-1)^{n-r}\frac{r\,!}{(n-r)\,!}(xy)^{n-r}L_{r}^{(\alpha-r)}(x)L_{r}^{(\beta-r)}(y) \tag{9}$$

$$=\sum_{j+k\le n}(-1)^{j+k}\binom{\alpha}{n-j}\binom{\beta}{n-k}\frac{(n-j)\,!\,(n-k)\,!}{j\,!\,k\,!\,(n-j-k)\,!}x^{j}y^{k}.$$

If we replace α by $\alpha+n$ and β by $\beta+n$, (9) becomes

$$\sum_{r=0}^{n}(-1)^{n-r}\frac{r\,!}{(n-r)\,!}(xy)^{n-r}L_{r}^{(\alpha+n-r)}(x)L_{r}^{(\beta+n-r)}(y) \tag{10}$$

$$=(\alpha+1)_{n}(\beta+1)_{n}\sum_{j+k\le n}(-1)^{j+k}\frac{x^{j}y^{k}}{j\,!\,k\,!\,(n-j-k)\,!\,(\alpha+1)_{j}(\beta+1)_{k}}\,.$$

We shall no givwe a direct proof of (10) valid for arbitrary α, β. By (4) the left hand side of (10) is equal to

$$\sum_{r=0}^{n} (-1)^{n-r} \frac{r!}{(n-r)!} (xy)^{n-r} \sum_{j=0}^{r} \binom{\alpha+n}{j} \frac{(-x)^{r-j}}{(r-j)!} \sum_{k=0}^{r} \binom{\beta+n}{k} \frac{(-y)^{r-k}}{(r-k)!}$$

$$= \sum_{j,k=0}^{n} (-1)^{j+k} \frac{(\alpha+1)_n(\beta+)_n}{j!\,k!\,(\alpha+1)_{n-j}(\beta+1)_{n-k}} x^{n-j} y^{n-k}$$

$$\cdot \sum_{r=0}^{n} (-1)^{n-r} \frac{r!}{(n-r)!} \frac{1}{(r-j)!\,(r-k)!}$$

$$= \sum_{j,k=0}^{n} (-1)^{j+k} \frac{(\alpha+1)_n(\beta+1)_n}{(n-j)!\,(n-k)!\,(\alpha+1)_j(\beta+1)_k}$$

$$\cdot \sum_{r=0}^{n} (-1)^{n-r} \frac{r!}{(n \quad r)!} \frac{1}{(r-n+j)!\,(r-n+k)!},$$

where in the inner sum

$$r \geq \max(n-j,\ n-k).$$

If $j \geq k$ the inner sum is equal to

$$\sum_{r=0}^{k} (-1)^{k-r} \frac{(r+n-k)!}{r!\,(k-r)!\,(r+j-k)!} = (-1)^k \frac{(n-k)!}{k!\,(j-k)!} \sum_{r=0}^{k} \frac{(-k)_r(n-k+1)_r}{r!\,(j-k+1)_r}$$

$$= (-1)^k \frac{(n-k)!}{k!\,(j-k)!} \frac{(j-n)_k}{(j-k+1)_k} = \frac{(n-j)!\,(n-k)!}{j!\,k!\,(n-j-k)!},$$

provided $j+k \leq n$; if $j+k > n$ the sum vanishes. If $j \leq k$ it is evident that the same sum is obtained. Hence the left side of (10) reduces to

$$\sum_{j+k \leq n} (-1)^{j+k} \frac{(\alpha+1)_n(\beta+1)_n x^j y^k}{j!\,k!\,(n-j-k)!\,(\alpha+1)_j(\beta+1)_k}, \tag{11}$$

so that we have proved (10).

3. We recall that the JACOBI polynomial $P_n^{(\alpha,\beta)}(x)$ of degree n is defined by

$$P_n^{(\alpha,\beta)}(x) = \sum_{r=0}^{n} \binom{n+\alpha}{n-r}\binom{n+\beta}{r}\left(\frac{x-1}{2}\right)^r\left(\frac{x+1}{2}\right)^{n-r}.$$

Thus the double sum is (11) is equal to

$$\sum_{r=0}^{n} (-1)^r \frac{(\alpha+1)_n(\beta+1)_n}{(n-r)!\,(\alpha+1)_r(\beta+1)_r} \sum_{j=0} \frac{(\alpha+1)_r(\beta+1)_r}{j!\,(r-j)!\,(\alpha+1)_j(\beta+1)_{r-j}}\, x^j y^{r-y}$$

$$= \sum_{r=0}^{n} (-1)^r \frac{(\alpha+1)_n(\beta+1)_n}{(n-r)!\,(\alpha+1)_r(\beta+1)_r} \sum_{j=0}^{r} \binom{r+\alpha}{r-j}\binom{r+\beta}{j} x^j y^{r-j}$$

$$= \sum_{r=0}^{n} (-1)^r \frac{(\alpha+1)_n(\beta+1)_n}{(n-r)!\,(\alpha+1)_r(\beta+1)_r} (y-x)^r P_r^{(\alpha,\beta)}\left(\frac{y+x}{y-x}\right).$$

Hence (10) can be written as

$$(12)\qquad \sum_{r=0}^{n} (-1)^{n-r} \frac{r!}{(n-r)!} (xy)^{n-r} L_r^{(\alpha+n-r)}(x) L_r^{(\beta+n-r)}(y)$$

$$= \sum_{r=0}^{n} (-1)^r \frac{(\alpha+1)_n(\beta+1)_n}{(n-r)!\,(\alpha+1)_r(\beta+1)_r} (y-x)^r P_r^{(\alpha,\beta)}\left(\frac{y+x}{y-x}\right)$$

provided $x \neq y$. When $x = y$ we get

$$(13)\qquad \sum_{r=0}^{n} (-1)^{n-r} \frac{r!}{(n-r)!} x^{2n-2r} L_r^{(\alpha+n-r)}(x) L_r^{(\beta+n-r)}(x)$$

$$= \sum_{r=0}^{n} (-1)^r \frac{(\alpha+1)_n(\beta+1)_n(\alpha+\beta+r+1)_r}{r!\,(n-r)!\,(\alpha+1)_r(\beta+1)_r} x^r.$$

4. It is evident that the transformations (10), (12), (13) are applicable to the expression $\Delta_n^{(\lambda)}$ defined by (7). In particular, using

(10), (6) becomes

$$(14)\ (\lambda)_{n+1}L_n^{(\lambda)}(x^2+y^2-2xyz)=\sum_{r=0}^{n}(xy)^{n-r}(\lambda+n-r)C_{n-r}^{(\lambda)}(z)\{(\lambda+n-r+1)_r\}^2$$

$$\cdot\sum_{j+k\leq r}(-1)^{j+k}\frac{x^{2j}y^{2k}}{j\,!\,k\,!\,(r-j-k)\,!\,(\lambda+n-r+1)_j(\lambda+n-r+1)_k}.$$

Since

$$\Gamma(\lambda-1)(xt)^{-\lambda}e^{t^2}J_\lambda(2xt)=\sum_{n=0}^{\infty}\frac{L_n^{(\lambda)}(x^2)t^{2n}}{(\lambda+1)_n},$$

it follows that (14) is equivalent to

$$(15)\quad \lambda\Gamma(\lambda+1)(wt)^{-\lambda}e^{t^2}J_\lambda(2wt)=\sum_{n=0}^{\infty}t^{2n}\sum_{r=0}^{n}(xy)^{n-r}(\lambda+n-r)C_{n-r}^{(\lambda)}(z)$$

$$\sum_{j+k\leq r}(-1)^{j+k}\frac{x^{2j}y^{2k}}{j\,!\,k\,!\,(r-j-k)\,!\,(\lambda+1)_{n-r+j}(\lambda+1)_{n-r+k}},$$

where

$$w^2=x^2+y^2-2xyz.$$

Now the right side of (15) is equal to

$$\Gamma^2(\lambda+1)\sum_{r=0}^{\infty}t^{2r}\sum_{n=0}^{\infty}(xyt^2)^n(\lambda+n)C_n^{(\lambda)}(z)$$

$$\sum_{j+k\leq r}(-1)^{j-k}\frac{x^{2j}y^{2k}}{j\,!\,k\,!\,(r-j-k)\,!\,\Gamma(\lambda+n+j+1)\Gamma(\lambda+n+k+1)}$$

$$=\Gamma^2(\lambda+1)\sum_{n=0}^{\infty}(\lambda+n)C_n^{(\lambda)}(z)$$

$$\cdot\sum_{j,k=0}^{\infty}(-1)^{j+k}\frac{(xt)^{2j+n}(yt)^{2k+n}}{j\,!\,k\,!\,\Gamma(\lambda+n+j+1)\Gamma(\lambda+n+k+1)}\cdot\sum_{r=0}^{\infty}\frac{t^{2r}}{r\,!}$$

$$=\Gamma^2(\lambda+1)(xyt^2)^{-\lambda}e^{t^2}\sum_{n=0}^{\infty}(\lambda+n)C_n^{(\lambda)}(z)\cdot J_{\lambda+n}(2xt)J_{\lambda+n}(2xt).$$

Thus (15) becomes

$$\frac{J_\lambda(2wt)}{(wt)^\lambda} = \Gamma(\lambda) \sum_{n=0}^{\infty} (\lambda + n) C_n^{(\lambda)}(z) J_{\lambda+n}(2xt) J_{\lambda+n}(2yt), \tag{16}$$

the addition theorem for $J_\lambda(t)$ [6, p. 363]. Since these steps are reversible we have evidently proved that (6) is equivalent to (16). In particular BATEMAN'S formula (3) is equivalent to the addition theorem for $J_0(\lambda)$.

5. It may be of interest to note that if in the familiar generating functions

$$\sum_{n=0}^{\infty} L_n^{(\alpha)}(x) t^n = (1 - t)^{-\alpha-1} \exp\left(-\frac{xt}{1-t}\right),$$

$$\sum_{n=0}^{\infty} L_n^{(\alpha-n)}(x) u^n = (1 + u)^n e^{-xu}$$

we take $u = t/(1 - t)$, we obtain the following pair of formulas:

$$L_n^{(\alpha-n)}(x) = \sum_{r=0}^{n} (-1)^{n-r} \binom{n}{r} L_r^{(\alpha)}(x), \tag{17}$$

$$L_n^{(\alpha)}(x) = \sum_{r=0}^{n} \binom{n}{r} L_r^{(\alpha-r)}(x). \tag{18}$$

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