Bollettino Unione Matematica Italiana

P. L. Sharma

On absolute logarithmic summability of double Fourier series.

Bollettino dell'Unione Matematica Italiana, Serie 3, Vol. **16** (1961), n.1, p. 19–23.

Zanichelli

<http://www.bdim.eu/item?id=BUMI_1961_3_16_1_19_0>

Bollettino dell'Unione Matematica Italiana, Zanichelli, 1961.

On absolute logarithmic summability of double Fourier series.

Nota di P. L. SHARMA (India) (*)

Summary. - *The author defines absolute logarithmic summability for double series and studies a problem relating absolute summability factors for double* FOURIER *serie.*

1. DEF. A. – The double series $\Sigma a_{m,n}$ is said to be absolutely summable $(R, \lambda, \mu; 1, 1)$ or summable $|R, \lambda, \mu; 1, 1|$ when the integral

$$\int_0^\infty \int_0^\infty \left| \frac{\partial^2 \Phi(x, y)}{\partial x \partial y} \right| dxdy < \infty,$$

where

$$\Phi(x, y) = x^{-1} y^{-1} \sum_{\lambda_m < x} \sum_{\mu_n < y} a_{m,n}(x - \lambda_m)(y - \mu_n)$$

as $(x, y) \to (+\infty, +\infty)$.

This is known [3].

DEF. B. – Let $|\lambda(\omega)|$ and $|\mu(\Omega)|$ be continuous, differentiable and monotonic increasing in (A, ∞) and (B, ∞) respectively, where A and B are some positive numbers, and let $\lambda(\omega)$ and $\mu(\Omega)$ tend towards infinity with ω and Ω.

Suppose

$$c(\omega, \Omega) = \sum_{n < \omega} |\lambda(\omega) - \lambda(n)| \sum_{m < \Omega} |\mu(\Omega) - \mu(m)| \, a_{m,n},$$

then the series $\Sigma a_{m,n}$ is said to be summable $|R, \lambda(n), \mu(m); 1,, 1|$ if

$$\int_A^\infty \int_B^\infty \left| d \left\{ \frac{c(\omega, \Omega)}{\lambda(\omega)\mu(\Omega)} \right\} \right| < \infty$$

i.e., if

$$\int_A^\infty \int_B^\infty \frac{\lambda'(\omega)\mu'(\Omega)}{|\lambda(\omega)\mu(\Omega)|^2} \left| \sum_{n \leq \omega} \sum_{m \leq \Omega} \lambda(n)\mu(m) a_{m,n} \right| d\omega d\Omega < \infty,$$

$$\int_A^\infty \frac{\lambda'(\omega)}{|\lambda(\omega)|^2} \left| \sum_{n \leq \omega} \lambda(n) a_{0,n} \right| d\omega < \infty, \quad \int_B^\infty \frac{\mu'(\Omega)}{|\mu(\Omega)|^2} \left| \sum_{m \leq \Omega} \mu(m) a_{m,0} \right| d\Omega < \infty,$$

(*) Pervenuta alla Segreteria dell'U. M. I. il 15 gennaio 1961.

The consistency theorems for general absolute RIESZ summability of double series are known [3].

2. Let $f(x, y)$ be an even function integrable in the LEBESGUE sense over the square $Q(-\pi, -\pi; \pi, \pi)$ is doubly periodic with period 2π in each variable. The double FOURIER series associated with the function $f(x, y)$ at the origin is

$$(2.1)\qquad \sum_{m,n=0} a_{m,n},$$

where

$$a_{m,n} = \frac{4}{\pi^2}\iint_Q f(u, v)\cos mu \cos nv \, dv du.$$

We write

$$(2.2)\quad \Phi(u, v) = \frac{1}{4}\{f(x+u, y+v) + f(x-u, y+v) + f(x+u, y-v) + f(x-u) \; y-v) - 4s\}$$

$$\Phi_{\alpha,\beta}(u, v) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^u\int_0^v (u-t)^{\alpha-1}(v-\theta)^{\beta-1}\Omega(t, \theta)dt d\theta \quad (\alpha, \beta > 0)$$

$$\Phi_{0,0}(u, v) = \Phi(u, v); \; \Phi_{\alpha,0}(u, 0) = \frac{1}{\Gamma(\alpha)}\int_0^u (u-t)^{\alpha-1}\Phi(t, v)dt,$$

$$(2.3)\qquad \Phi_{\alpha,\beta}(u, v) = u^{-\alpha}v^{-\beta}\Gamma(\alpha+1)\Gamma(\beta+1)\Phi_{\alpha,\beta}(u, v), \qquad (\alpha, \beta > 0$$

$$\Phi_{0,0}(u, v) = \Phi(u, v), \; \Phi_{\alpha,0}(u, v) = u^{-\alpha}\Phi_{\alpha,0}(u, v)\,\Gamma(\alpha+1),$$

$$(2.4)\qquad \xi(\omega, t) = \sum_{n\leq\omega} e^{(\log n)^\delta}(\log n)^{-1}\cos nt,$$

$$2.5)\qquad \eta(\omega, t) = \sum_{n\leq\omega} e^{(\log n)^\delta}(\log n)^{-1} n^{-1}\sin nt,$$

$$(2.6)\qquad g(\omega, u) = \frac{1}{\Gamma(1-\alpha)}\int_u^\pi (t-u)^{-\alpha}\xi(\omega, t)dt, \qquad 0 \leq u \leq \pi,$$

$$(2.7)\qquad G(\omega, u) = \frac{1}{\Gamma(1-\alpha)}\int_0^\alpha v^\alpha \frac{d}{dv} g(\omega, v)dv, \qquad 0 \leq u \leq \pi.$$

The object of this paper is to prove the following theorem:

THEOREM. - *If* [see HARDY ; 1]

$$\frac{\partial^2\Phi_{\alpha,\beta}(u,\ v)}{\partial u\partial v},\qquad \frac{\partial\Phi_{\alpha,0}(u,\ v)}{\partial u},\qquad \frac{\partial\Phi_{0,\beta}(u,\ v)}{\partial v}$$

is finite and of a constant sign in the rectangle $(0,\ 0\,;\ \pi,\ \pi)$ *then the double series*

$$\sum_2^\infty\sum_2^\infty \frac{a_{m,n}}{\log m\log n}$$

is summable $|R,\ e^{(\log n)^\tau},\ e^{(\log n)^\delta};\ 1,\ 1|$ *where* $0<\alpha<1,\ 0<\beta<1$ *and* $\tau=1+\dfrac{1}{\alpha},\ \delta=1+\dfrac{1}{\beta}$.

3. We need the following inequalities for the proof of our theorem.

$$\xi(\omega,\ t)=O\ \{e^{(\log\omega)^\delta}\omega(\log\omega)^{-\delta}\}\tag{3.1}$$
$$=O\ \{e^{(\log\omega)^\delta}t^{-1}(\log\omega)^{-1}\}$$

$$g(\omega,\ v)=O\ \{e^{(\log\omega)^\delta}\omega^\beta(\log\omega)^{-\delta}\}\tag{3.2}$$
$$=O\ \{e^{(\log\omega)^\delta}\omega^{-1+\beta}v^{-1}(\log\omega)^{-1}\}$$

$$G(\omega,\ v)=O\ \{e^{(\log\omega)^\delta}\omega^\beta v^\beta(\log\omega)^{-\delta}\}\tag{3.3}$$
$$=O\ \{e^{(\log\omega)^\delta}\omega^{-1+\beta}v^{-1+\beta}\log\omega)^{-1}\}$$

$$J=\int_2^\infty\delta\omega^{-1}(\log\omega)^{\delta-1}e^{-(\log\omega)^\delta}\,|\,G(\omega,\ v)\,|\,d\omega<\infty.\tag{3.4}$$

The inequalities (3.1), (3.2), (3.3) and (3.4) are known [2].

PROOF OF THEOREM. - To prove the theorem it is enough to show that when

$$\tau=1+\frac{1}{\alpha},\qquad \delta=1+\frac{1}{\beta},$$

$$I=\int_2^\infty\int_2^\infty\tau\Omega^{-1}(\log\Omega)^{\tau-1}e^{-(\log\Omega)^\tau}\delta\omega^{-1}(\log\omega)^{\delta-1}e^{-(\log\omega)^\delta}$$
$$|\,p(\Omega,\ \omega)\,|\,d\Omega d\omega<\infty,$$

where

$$p(\Omega,\ \omega) \sum_{n \leq \omega} \sum_{m \leq \Omega} e^{(\log m)^{\tau}} (\log m)^{-1} e^{(\log n)^{\delta}} (\log n)^{-1} a^{m,n} \qquad (\tau,\ \delta > 0).$$

We have

$$\frac{1}{4}\pi^2 p(\Omega,\ \omega) = \int_0^{\pi}\int_0^{\pi} \Phi(u,\ v) \frac{d}{du} \eta(\Omega,\ u) \frac{d}{du} \eta(\omega,\ v) du dv$$

Applying integration by parts for double integral [4], we have

$$\frac{1}{4}\pi^2 p(\Omega,\ \omega) = \Big\{ \Phi_{\alpha,\beta}(\pi,\ \pi) g(\Omega,\ \pi) g(\omega,\ \pi)$$

$$- g(\Omega,\ \pi) \int_0^{\pi} \Phi_{\alpha,\beta}(\pi,\ v) \frac{d}{dv} g(\omega,\ v) dv$$

$$- g(\omega,\ \pi) \int_0^{\pi} \Phi_{\alpha,\beta}(u,\ \pi) \frac{d}{du} g(\Omega,\ u) du$$

$$+ \int_0^{\pi}\int_0^{\pi} \Phi_{\alpha,\beta}(u,\ v) \frac{d}{du} g(\Omega,\ u) \frac{d}{dv} g(\omega,\ v) du dv \Big\}.$$

Using the inequalities (3.1), (3.2) and (3.3), we have,

$$\frac{\pi^2}{4} p(\Omega,\ \omega) = O\Big\{ e^{(\log \Omega)^{\tau}} \Omega^{-1+\alpha} (\log \Omega)^{-1} e^{(\log \omega)^{\delta}} \omega^{-1+\beta} (\log \omega)^{-1} \Big\}$$

$$+ O\Big\{ e^{(\log \Omega)^{\tau}} \Omega^{-1+\alpha} (\log \Omega)^{-1} \int_0^{\pi} \left| \frac{\partial \Phi_{\alpha,\beta}(\pi,\ v)}{\partial v} \right| \left| G(\omega,\ v) \right| du \Big\}$$

$$+ O\Big\{ e^{(\log \omega)^{\delta}} \omega^{-1+\beta} (\log \omega)^{-1} \int_0^{\pi} \left| \frac{\partial \Phi_{\alpha,\beta}(u,\ \pi)}{\partial u} \right| \left| G(\Omega,\ u) \right| du \Big\}$$

(3.5)
$$+ O\Big\{ \int_0^{\pi}\int_0^{\pi} \left| \frac{\partial^2 \Phi_{\alpha,\beta}(u,\ v)}{\partial u \partial v} \right| \left| G(\Omega,\ u \right| \left| G(\omega,\ v) \right| du dv \Big\}.$$

By the hypothesis of the theorem, we have,

$$\int_0^\pi \int_0^\pi \left| \frac{\partial^2 \Phi_{\alpha,\beta}(u, v)}{\partial u \partial v} \right| du dv < \infty, \tag{3.6}$$

$$\int_0^\pi \left| \frac{\partial \Phi_{\alpha,j}(u, \pi)}{\partial u} \right| du < \infty, \quad \int_0^\pi \left| \frac{\partial \Phi_{i,\beta}(\pi, v)}{\partial v} \right| dv < \infty. \tag{3.7}$$

Collecting (3.4), (3.5), (3.6) and (3.7), we get

$$|I| < \infty.$$

Hence the Theorem is proved.

I am indebted to Professor M. L. MISRA for his kind interest and guidance in the preparation of this paper.

REFERENCES

[1] HARDY, G. H., *On double Fourier series, and specially those which represent the double Zeta function with real and incommensurable parameters*, « Quart. Journ. Maths. » vol. 37, (1906), 53-79.

[2] MOHANTY, R. N. and MISRA, B., *On absolute logarithmic summability of a sequence related to Fourier series*, « Tohaku Math. Jour. », vol 6, (1954), 5-12.

[3] OBRECHKOFF, N., *Sur la sommation des series multiples de Dirichlet, et des series semblables*, « Ann. Univ. Sofia. Fac. Sci. Fivre 1, Math. Phys », vol. 36, (1940), 1-145.

[4] SHARMA, P. L. - *On harmonic summability of double Fourier series* « Proc. Amer. Math. Soc. », vol. 9, (1958), 979-986.