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On the Bernoulli Polynomial.

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Summary. - *The integral and other properties, of the Bernoullian polynomial of order* n *defined by* RAABE [4], *have bren considered.*

In a recent paper [1] MORDELL remarked that there seem to be no simple formulae for the integral of the product of three or more BERNOULLI polynomials.

L. CARLITZ [2] considered the integral of the product of any number of BERNOULLI polynomials defined by

$$\frac{te^{xt}}{e^t - 1} = \sum_{m=0}^{\alpha} Bm(x) \frac{t^m}{m} .$$

Here *I* shall consider the BERNOULLI polynomials defined by

$$\frac{t(e^{xt} - 1)}{e^t - 1} = \sum_{1}^{\alpha} \varphi_n(x) \frac{t^n}{n^n}$$

I shall give simpler method of integration of the product of any number of BERNOULLI polynomials and also the product of Bernomial and any other polynomial. *I* shall consider some properties of BERNOULLI Polynomial also.

Now

$$(1) \qquad t\,\frac{e^{xt} - 1}{e^t - 1} = \sum_{n=1}^{\alpha} \varphi_n(x) \frac{t^n}{n^n}$$

(*) Pervenuta alla Segreteria dell' U. M. I. il 15 ottobre 1960.

then

$$\sum_{n=1}^{\alpha} \varphi_n(x+y)\,\frac{t^n}{n^n} = \frac{te^{(x+y)t}-1}{e^t-1}$$

$$= e^{yt}\,t\,\frac{(e^{xt}-1)}{e^t-1} + \frac{t(e^{yt}-1)}{e^t-1} = \sum_{0}^{\alpha} \frac{y^m}{L^m}\,t^m \sum_{1}^{\alpha} \varphi_n(x)\,\frac{t^n}{n^n}$$

$$+ \sum_{1}^{\alpha} \varphi_n(y)\,\frac{t^n}{n^n}\,.$$

Hence,

$$\text{(2)} \qquad \varphi_n(x+y) = \varphi_n(y) + \sum_{\gamma=0}^{\alpha} nC_\gamma y^\gamma \varphi_{n-\gamma}(x)\,.$$

Again

$$t(e^{xt}-1) = (e^t-1)\sum_{n=1}^{\alpha} \varphi_n(x)\,\frac{t^n}{n^n}$$

$$\sum_{n=1}^{\alpha} \frac{x^n\,t^{n+1}}{t^n} = \sum_{1}^{\alpha} \frac{t^m}{n^m} \sum_{n=1}^{\alpha} \varphi_n(x)\,\frac{t^n}{n^n}\,.$$

Hence

$$\text{(3)} \qquad x^n = \varphi_n(x) + \frac{n}{2}\,\varphi_{n-1}(x) + \frac{n(n-1)}{2\cdot 3}\,\varphi_{n-2}(x) + \dots$$

Whittakar [3] proved that

$$\varphi_n(x) = x^n - \frac{n}{2}\,x^{n-1} + nC_2B_1x^{n-2}\dots$$

where $B_1B_2\dots$ are BERNOULLI's number, the last term will contain x or x^2 according as n is odd or even.

Hence,

$$\text{(4)} \qquad \int_0^1 x^m\varphi_n(x) = \frac{1}{m+n+1} - \frac{n}{2(m+n)} +$$

$$\frac{nC_2B_1}{n+m-1} + \dots = An,\ m \text{ (say)}.$$

Hence,

$$\text{(5)} \qquad \int_0^1 \varphi_n^2(x)dx = An,\ n - \frac{n}{2}\,An-1,\ n + {}^nC_2B_1An-2,\ n +$$

by substituting the values of $\varphi_n(x)$ in the left hand side of (4) and utilising (3).
Again,

(6) $$\int_0^1 \varphi_m(x)\varphi_n(x)dx - Am,\ n - \frac{m}{2}\ Am - 1.\ n$$

$$+ mC_2B_1Am - 2,\ n -$$

Again

$$\int_0^1 x^{m+r}\varphi_n(x)dx - Am + r,\ n$$

Hence

(7) $$\int_0^1 x^m\varphi_n(x)\varphi_r(x)dx$$

$$\equiv Am + r,\ n - \frac{r}{2}\ Am + r - 1.\ n$$

$$+ rC_2B_1Am + r - 2,\ n \ldots = Cm,\ rn \text{ (say)}$$

Hence,

(8) $$\int_0^1 \varphi_m(y)\,\varphi_r(y)\,\varphi_n(y)\ dy$$

$$= Cm,\ r^n\ \frac{m}{2}\ Cm - 1,\ r_n + mC_2B_1\ Cm = 2,\ r_n \ldots$$

Again

(9) $$\int_0^1 P_m(x)\,\varphi_n(x)dx = \sum_{r=0}^{m} \frac{(-1)^r L^{2m-2r}}{2^m L^r\ L^{m-r}\ L^{m-2r}}\ Am - 2 + r,\ n$$

where $P_m(x)$ is Legendri's polynomial.

Differentiating (1) with respect to x we have

$$\sum_1^\infty \varphi_n{}^1(x)\ \frac{t^n}{n^n} = \frac{t^2e^{xt}}{e^t - 1} = t\left[\frac{t(e^{xt} - 1)}{e^t - 1} + \frac{t}{e^t - 1}\right]$$

$$= \sum_1^\infty \varphi_n(x)\ \frac{t^{n+1}}{n''} + t\left[1 - \frac{1}{2}\ t + \sum_{n-1}^\infty (-1)^{n-1}B_n\ \frac{t^{2n}}{n(2n)}\right]$$

where Bn is BERNOULLI'S number.

Hence,

(10) $$\varphi_{2n}'(x) = 2n\,\varphi(2n - 1(x) \text{ if } n)2$$

(11) $$\varphi'(2n + 1(x) = (2n + 1)\,\varphi_{2n}(x) + (-1)^{n-1}(2n + 1) \times B_n \ldots .$$

Now

$$(u + v) \sum_{m=0}^{\infty} \sum_{n=0}^{\alpha} \varphi_m(x)\varphi_n(x) \frac{u^m v^n}{n^m\, n^n}$$

$$= \frac{(u + v)[e^{ux} - 1]\,[e^{vx} - 1]uv}{(e^u - 1)\,(e^v - 1)}$$

$$= (u + v) \frac{[e^{x(u+v)} - 1]}{e^{u+v} - 1} \left[uv + \frac{uv}{e^u - 1} + \frac{uv}{e^v - 1}\right]$$

$$- \frac{(u + v)uv}{e^v - 1} \frac{e^{ux} - 1}{e^u - 1} - \frac{(u + v)uv}{e^u - 1} \frac{e^{vx} - 1}{e^v - 1}$$

$$= \sum_0^{\alpha} \varphi_m(x) \frac{(u + v)^m}{L^m} \sum_0^{\alpha} \frac{B_r}{n^{2r}} (u^{2r}v + uv^{2r}) \times$$

$$(-1)^{r-1}\Big] - (u + v) \left[\sum_0^{\alpha} \varphi_m(x) \frac{v^m}{L^m} \right\{ 1 - \frac{u}{2} -$$

$$\sum_1^{\alpha} \frac{(-1)^n B_n u^{2n}}{L^{2n}} \Big\}\Big] - (u + v) \left[\sum_0^{\alpha} \varphi_m(x) \frac{u^m}{n^m} \right.$$

$$\left\{ 1 - \frac{v}{2} - \sum_1^{\alpha} (-1)^n B_n \frac{v^{2n}}{n^{2n}} \right\}\Big) .$$

Where $Bo = -1$.

Equating the coefficient of $\frac{u^{2m}}{n^{2m}} \frac{v^{2n}}{n^{2n}}$ we have

$$2m\varphi 2m - 1(x)\varphi 2n(x) + 2n\varphi 2n = 1(x)\varphi_{2m}(x)$$

$$= \sum_r \left[\binom{2m}{2r} 2n + \binom{2m}{2r} 2m\right] (-1)^{r-1} Br\varphi 2m + 2n - 2r - 1(x)$$

$$+ (-1)^n B_n 2m\varphi_{2m-1}(x) + (-1)^m B_m 2n\varphi_{2n-1}(x)$$

Hence

$$\varphi'_{3m}(x)\,\varphi_{2n}(x) + \varphi'_{2n}(x)\,\varphi_{2m}(x)$$

$$= \sum_r \left[\binom{2m}{2r} 2n + \binom{2n}{2r} 2m \right] (-1)^{r-1} B_r \varphi' \frac{2m+2n-2r}{2m+2n-2r}(x)$$

$$+ (-1)^n B_n \varphi'_{2m}(x) + (-1)^m B_m \varphi'_{2n}(x)\,.$$

Hence,

$$\varphi_{2m}(x) \qquad \varphi_{2n}(x)$$

$$= \sum_r \left[\binom{2m}{2r} 2n + \binom{2n}{2r} 2m \right] (-1)^{n-1} B_r \varphi \frac{2m+2n-2r}{2m+2n-2r}(x)$$

$$+ (-1)^n B_n \varphi_{2m}(x) + (-1)^m B_m \varphi_{2n}(x) + C\,.$$

Putting $x = 0$, we have $C = 0$ since $\varphi_n(x) = 0$ if $x = 0$. Hence

$$\varphi_{2m}(x) \qquad \varphi_{2n}(x)$$

$$= \sum_r \left[\binom{2m}{2r} 2n + \binom{2n}{2r} 2m \;\; (-1)^{r-1} B_r \right.$$

$$\varphi \frac{2m+2n-2r}{2m+2n-2r}(x) + (-1)^n B_n \varphi_{2m}(x)$$

$$+ (-1)^m B_m \varphi_{2n}(x)\,.$$

Similarly, we can find

$$\varphi_{2m+1}(x) \qquad \varphi_{2n}(x)$$

and

$$\varphi_{2m+1}(x) \qquad \varphi_{2n}(x)\,.$$

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[4] RAABE, Journal für Math. XLII (185) p. 348.