Bollettino Unione Matematica Italiana

S. K. Chatterjea

Some formulae of symbolic calculus for the complete elliptic integrals of the first and second kind.

Bollettino dell'Unione Matematica Italiana, Serie 3, Vol. **15** (1960), n.4, p. 471–478.

Zanichelli

<http://www.bdim.eu/item?id=BUMI_1960_3_15_4_471_0>

Bollettino dell'Unione Matematica Italiana, Zanichelli, 1960.

SEZIONE SCIENTIFICA

BREVI NOTE

Some formulæ of symbolic calculus for the complete elliptic integrals of the first and second kind.

Nota di S. K. Chatterjea (a Calcutta) (*)

Summary. - *The first part of this paper deals with certain formulae (of symbolic calculus) relating to the complete elliptic integrals of the first and second kind. In the second part a decent number of classical properties of these integrals have been verified by alternative but independent methods.*

1. If

$$g(s)=\int_0^\infty f(x)\, x^{s-1}\, dx\,;$$

then $g(s)$ is called the Mellin transform of $f(x)$. We shall denote this integral symbolically as

$$g(s)=M[f(x);\ s].$$

In Mellin transform we have [1]

$$\int_0^\infty e^{-\alpha x}\,\Phi(\beta,\ \rho;\ \lambda x)\, x^{s-1} dx=\frac{\Gamma(s)}{\alpha^s}\,F(\beta,\ s;\ \rho;\ \lambda\alpha^{-1}), \tag{1.1}$$

$$(s>0,\ \alpha>0).$$

Putting $\beta=\frac{1}{2}$, $s=\frac{1}{2}$, $\rho=1$ in (1.1) we get

(*) Pervenuta alla Segreteria dell' U.M.I. l' 11 ottobre 1960.

(1.2)
$$\int_0^\infty e^{-\alpha x}\,\Phi\left(\frac{1}{2},\ 1;\ \lambda x\right) x^{-\frac{1}{2}}\,dx = \frac{\Gamma\left(\frac{1}{2}\right)}{\alpha^{\frac{1}{2}}}\,F\left(\frac{1}{2},\ \frac{1}{2};\ 1;\ \lambda\alpha^{-1}\right).$$

Now the LAGUERRE functions $L_\nu^{(\alpha)}(x)$, for unrestricted values of ν, are an alternative notation [2] for Whittaker function:

$$L_\nu^{(\alpha)}(x) = \frac{\Gamma(\alpha+\nu+1)}{\Gamma(\alpha+1)\,\Gamma(\nu+1)}\cdot x^{-\frac{1}{2}\alpha-\frac{1}{2}}\,e^{\frac{1}{2}x}\,M_{\frac{1}{2}\alpha+\nu+\frac{1}{2},\ \frac{1}{2}\alpha}(x)$$

(1.3)
$$= \frac{\Gamma(\alpha+\nu+1)}{\Gamma(\alpha+1)\,\Gamma(\nu+1)}\cdot\Phi(-\nu,\alpha+1;\ x)$$

we have

(1.4)
$$\Phi\left(\frac{1}{2},\ 1;\ \lambda x\right) = L_{-\frac{1}{2}}^{(0)}(\lambda x).$$

Again the complete elliptic integrals of the first and second kind viz.,

$$K(k) = \int_0^{\pi/2}\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}\,;\quad |\,k\,| < 1$$

and

$$E(k) = \int_0^{\pi/2}\sqrt{1-k^2\sin^2\theta}\,d\theta\,;$$

are alternative notations [3] for hypergeometric function of GAUSS:

(1.5)
$$K(k) = \frac{\pi}{2}\,F\left(\frac{1}{2},\ \frac{1}{2};\ 1;\ k^2\right)$$

and

(1.6)
$$E(k) = \frac{\pi}{2}\,F\left(-\frac{1}{2},\ \frac{1}{2};\ 1;\ k^2\right).$$

Thus we easily obtain from (1.2), (1.4) and (1.5)

(1.7)
$$\int_0^\infty e^{-\alpha x}\,L_{-\frac{1}{2}}^{(0)}(\lambda x)\,x^{-\frac{1}{2}}\,dx = \frac{2}{\sqrt{\pi\alpha}}\,K\left(\sqrt{\frac{\lambda}{\alpha}}\right).$$

In symbolic calculus, (1.7) can be written as

$$\frac{2}{\sqrt{\pi\alpha}} K\left(\sqrt{\frac{\lambda}{\alpha}}\right) = M\left[e^{-\alpha x} L^{(0)}_{-\frac{1}{2}}(\lambda x);\ \frac{1}{2}\right]. \tag{1.8}$$

In LAPLACE transform (1.7) reads like

$$\frac{2}{\sqrt{\pi\alpha}} K\left(\sqrt{\frac{\lambda}{\alpha}}\right) = L\left[L^{(0)}_{-\frac{1}{2}}(\lambda x)\cdot x^{-\frac{1}{2}};\ \alpha\right]; \tag{1.9}$$

where $L[f(x);\ s] = \int_0^\infty e^{-sx} f(x)\, dx.$

Since MELLIN transform can be expressed either as exponential FOURIER transform in the complex domain, or as combinations of LAPLACE transforms:

$$M[f(x);\ s] = F[f(e^x);\ is] = L[f(e^x);\ -s] + L[f(e^{-x});\ s]$$

we have

$$\begin{aligned}
\frac{2}{\sqrt{\pi\alpha}}\cdot K\left(\sqrt{\frac{\lambda}{\alpha}}\right) &= M\left[e^{-\alpha x} L^{(0)}_{-\frac{1}{2}}(\lambda x);\frac{1}{2}\right]\\
&= L\left[L^{(0)}_{-\frac{1}{2}}(\lambda x)\cdot x^{-\frac{1}{2}};\ \alpha\right]\\
&= F\left[e^{-\alpha e^x} L^{(0)}_{-\frac{1}{2}}(\lambda e^x);\ \frac{1}{2}\, i\right]\\
&= L\left[e^{-\alpha e^x} L^{(0)}_{-\frac{1}{2}}(\lambda e^x);\ -\frac{1}{2}\right] + L\left[e^{-\alpha e^{-x}} L^{(0)}_{-\frac{1}{2}}(\lambda e^{-x});\ \frac{1}{2}\right].
\end{aligned} \tag{1.10}$$

Next using $\beta = -\frac{1}{2}$, $s = \frac{1}{2}$, $\rho = 1$ in (1.1) and proceeding exactly in the same way we find

$$\int_0^\infty e^{-\alpha x} L^{(0)}_{\frac{1}{2}}(\lambda x) x^{-\frac{1}{2}}\, dx = \frac{2}{\sqrt{\pi\alpha}} E\left(\sqrt{\frac{\lambda}{\alpha}}\right). \tag{1.11}$$

Thus we have

$$\frac{2}{\sqrt{\pi\alpha}} \cdot E\left(\sqrt{\frac{\lambda}{\alpha}}\right) = M\left[e^{-\alpha x} L^{(0)}{}_{\frac{1}{2}}(\lambda x);\ \frac{1}{2}\right]$$

$$= L\left[L^{(0)}{}_{\frac{1}{2}}(\lambda x) \cdot x^{-\frac{1}{2}};\ \alpha\right]$$

(1.12)

$$= F\left[e^{-\alpha e^{x}} L^{(0)}{}_{\frac{1}{2}}(\lambda e^{x});\ \frac{1}{2} i\right]$$

$$= L\left[e^{-\alpha e^{x}} L^{(0)}{}_{\frac{1}{2}}(\lambda e^{x});\ -\frac{1}{2}\right] + L\left[e^{-\alpha e^{-x}} L^{(0)}{}_{\frac{1}{2}}(\lambda e^{-x});\ \frac{1}{2}\right].$$

The following analogous formulae of symbolic calculus for the complete elliptic integrals of the first and second kind were obtained by TOSCANO [4]

$$(1.13) \qquad L\left[x^{-\frac{1}{2}} e^{\frac{\nu x}{2}} I_0\left(\frac{\nu x}{2}\right);\ \lambda\right] = \frac{2}{\sqrt{\pi\lambda}} K\left(\sqrt{\frac{\nu}{\lambda}}\right),$$

and

$$(1.14) \qquad L\left[x^{-\frac{3}{2}} e^{\frac{\nu x}{2}} I_0\left(\frac{\nu x}{2}\right);\ \lambda\right] = -4\sqrt{\frac{\lambda}{\pi}}\, E\left(\sqrt{\frac{\nu}{\lambda}}\right).$$

2. Using the following integral-representations for the complete elliptic integrals of the first and second kind:

$$K\left(\sqrt{\frac{\lambda}{\alpha}}\right) = \frac{1}{2}\sqrt{\pi\alpha}\int_0^\infty e^{-\alpha x} L^{(0)}_{-\frac{1}{2}}(\lambda x)\, x^{-\frac{1}{2}}\, dx$$

$$E\left(\sqrt{\frac{\lambda}{\alpha}}\right) = \frac{1}{2}\sqrt{\pi\alpha}\int_0^\infty e^{-\alpha x} L^{(0)}{}_{\frac{1}{2}}(\lambda x)\, x^{-\frac{1}{2}}\, dx$$

we shall now obtain some known properties of such integrals.

First using $\alpha = 1$ and $\lambda = k^2$, we obtain

$$(2.1) \qquad K(k) = \frac{1}{2}\sqrt{\pi}\int_0^\infty e^{-x} L^{(0)}_{-\frac{1}{2}}(k^2 x)\, x^{-\frac{1}{2}}\, dx$$

$$(2.2) \qquad E(k) = \frac{1}{2}\sqrt{\pi}\int_0^\infty e^{-x} L^{(0)}{}_{\frac{1}{2}}(k^2 x)\, x^{-\frac{1}{2}}\, dx.$$

Next using $\alpha = \alpha^2$ and $\lambda = -\lambda^2$, we derive

$$(2.3) \qquad \frac{2}{\sqrt{\pi} \cdot \alpha} K\left(i \frac{\lambda}{\alpha}\right) = \int_0^\infty e^{-\alpha^2 x} L_{-\frac{1}{2}}^{(0)}(-\lambda^2 x)\, x^{-\frac{1}{2}}\, dx.$$

$$(2.4) \qquad \frac{2}{\sqrt{\pi} \cdot \alpha} E\left(i \frac{\lambda}{\alpha}\right) = \int_0^\infty e^{-\alpha^2 x} L^{(0)}_{\frac{1}{2}}(-\lambda^2 x)\, x^{-\frac{1}{2}}\, dx.$$

Now we know [2, p. 53]

$$(2.5) \qquad L_\nu^{(\mu)}(x) = \frac{\operatorname{Sin} \pi\nu}{\operatorname{Sin} \pi(\mu + \nu)} \cdot e^x L_{-\mu-\nu-1}^{(\mu)}(-x)$$

it follows from (2.3) and (2.5) that

$$\frac{2}{\sqrt{\pi} \cdot \alpha} K\left(i \frac{\lambda}{\alpha}\right) = \int_0^\infty e^{-(\alpha^2+\lambda^2)x} L_{-\frac{1}{2}}^{(0)}(\lambda^2 x) \cdot x^{-\frac{1}{2}}\, dx$$

$$= \int_0^\infty e^{-x} L_{-\frac{1}{2}}^{(0)}(\lambda^2 x) \cdot x^{-\frac{1}{2}}\, dx; \quad \textit{if} \quad \alpha^2 + \lambda^2 = 1.$$

$$= \frac{2}{\sqrt{\pi}} K(\lambda).$$

Thus we have

$$(2.6) \qquad K\left(i \frac{\lambda}{\alpha}\right) = \alpha\, \mathrm{K}(\lambda), \quad \text{where} \quad \alpha^2 + \lambda^2 = 1.$$

Again from (2.4) and (2.5) it follows that

$$(2.7) \qquad \begin{aligned} \frac{2}{\sqrt{\pi} \cdot \alpha} E\left(i \frac{\lambda}{\alpha}\right) &= \int_0^\infty e^{-(\alpha^2+\lambda^2)x} L_{-\frac{3}{2}}^{(0)}(\lambda^2 x) \cdot x^{-\frac{1}{2}}\, dx \\ &= \int_0^\infty e^{-x} L_{-\frac{3}{2}}^{(0)}(\lambda^2 x) \cdot x^{-\frac{1}{2}}\, dx; \text{ if } \alpha^2 + \lambda^2 = 1. \end{aligned}$$

Next we observe that [2, p. 56]

$$\text{(2.8)}\quad \int_0^\infty e^{-t}\, t^\lambda\, L_\nu^{(\mu)}(xt)\, dt = \frac{\Gamma(\lambda+1)\Gamma(\mu+\nu+1)}{\Gamma(\mu+1)\Gamma(\nu+1)} F(-\nu, \lambda+1; \mu+1; x)$$

$$|x| < 1, \ \lambda > -1.$$

Thus from (2.7) and (2.8) we have

$$\text{(2.9)}\qquad \frac{2}{\sqrt{\pi \cdot \alpha}} E\left(i\frac{\lambda}{\alpha}\right) = \Gamma\left(\frac{1}{2}\right) F\left(\frac{3}{2}, \frac{1}{2}; 1; \lambda^2\right).$$

Again we notice that

$$\text{(2.10)}\qquad F(a, b; c; x) = (1-x)^{c-a-b} F(c-a, c-b; c; x).$$

Thus finally from (2.9) and (2.10) we obtain

$$\frac{2}{\sqrt{\pi \cdot \alpha}} E\left(i\frac{\lambda}{\alpha}\right) = \sqrt{\pi} \cdot \frac{1}{(1-\lambda^2)} F\left(-\frac{1}{2}, \frac{1}{2}; 1; \lambda^2\right)$$

$$= \sqrt{\pi} \cdot \frac{1}{\alpha^2} \cdot \frac{2}{\pi} E(\lambda),$$

whence we have

$$\text{(2.11)}\qquad E\left(i\frac{\lambda}{\alpha}\right) = \frac{1}{\alpha} E(\lambda), \quad \text{where } \alpha^2 + \lambda^2 = 1.$$

Next we shall verify the result

$$\text{(2.12)}\qquad \frac{d}{dk}\{K(k)\} = \frac{E(k)}{kk'^2} - \frac{K(k)}{k}, \quad k^2 + k'^2 = 1.$$

To this end, we notice that

$$\frac{d}{d\alpha}\left\{\frac{2}{\sqrt{\pi\alpha}} K\left(\sqrt{\frac{\lambda}{\alpha}}\right)\right\} = -\int_0^\infty e^{-\alpha x} L_{-\frac{1}{2}}^{(0)}(\lambda x)\, x^{\frac{1}{2}}\, dx$$

(2.13)

$$\frac{2}{\sqrt{\pi\alpha}} \cdot \frac{d}{d\alpha}\left\{K\left(\sqrt{\frac{\lambda}{\alpha}}\right)\right\} = \frac{1}{\sqrt{\pi} \cdot \alpha^{3/2}} K\left(\sqrt{\frac{\lambda}{\alpha}}\right) - \int_0^\infty e^{-\alpha x} L_{-\frac{1}{2}}^{(0)}(\lambda x) \cdot x^{\frac{1}{2}}\, dx.$$

Now following the method of PINNEY [2, p. 56] one can easily obtain :

$$(2.14)\quad \int_0^\infty e^{-\alpha t}\, t^\lambda\, L_\nu^{(\mu)}(xt)dt = \frac{1}{\alpha^{\lambda+1}} \cdot \frac{\Gamma(\lambda+1)\Gamma(\mu+\nu+1)}{\Gamma(\mu+1)\Gamma(\nu+1)} F\left(-\nu, \lambda+1; \mu+1; \frac{x}{\alpha}\right)$$

$$|\, x \,| < 1,\ \lambda > -1.$$

It may be noted here that the formulae (1.7) and (1.11) can be easily verified from (2.14).

Thus we have from (2.13) and (2.14)

$$\frac{2}{\sqrt{\pi\alpha}} \cdot \frac{d}{d\alpha}\left\{ K\left(\sqrt{\frac{\lambda}{\alpha}}\right)\right\}$$

$$= \frac{1}{\sqrt{\pi} \cdot \alpha^{3/2}} K\left(\sqrt{\frac{\lambda}{\alpha}}\right) - \frac{1}{\alpha^{3/2}} \Gamma\left(\frac{1}{2}+1\right) F\left(\frac{1}{2}, \frac{3}{2}; 1; \frac{\lambda}{\alpha}\right)$$

$$= \frac{1}{\sqrt{\pi} \cdot \alpha^{3/2}} K\left(\sqrt{\frac{\lambda}{\alpha}}\right) - \frac{1}{\alpha^{3/2}} \cdot \frac{\sqrt{\pi}}{2} \cdot \frac{1}{\left(1-\frac{\lambda}{\alpha}\right)} \cdot \frac{2}{\pi} E\left(\sqrt{\frac{\lambda}{\alpha}}\right)$$

$$\frac{d}{d\alpha}\left\{ K\left(\sqrt{\frac{\lambda}{\alpha}}\right)\right\} = \frac{1}{2\alpha}\left[K\left(\sqrt{\frac{\lambda}{\alpha}}\right) - \frac{\alpha}{\alpha-\lambda} E\left(\sqrt{\frac{\lambda}{\alpha}}\right)\right].$$

Now using $\alpha = \frac{\lambda}{k^2}$, we easily obtain

$$\frac{d}{dk}\{K(k)\} = -\frac{2k\alpha^2}{\lambda} \cdot \frac{1}{2\alpha}\left[K(k) - \frac{\alpha}{\alpha-\lambda} E(k)\right]$$

$$= -\frac{\alpha k}{\lambda} K(k) + \frac{\alpha k}{\lambda} \cdot \frac{\alpha}{\alpha-\lambda} E(k)$$

$$= -\frac{K(k)}{k} + \frac{1}{k} \cdot \frac{1}{1-k^2} E(k)$$

$$= \frac{1}{kk'^2} E(k) - \frac{1}{k} K(k), \quad \text{where } k^2 + k'^2 = 1.$$

Lastly we shall deduce the property

$$(2.15)\qquad \frac{d}{dk}\{E(k)\} = \frac{1}{k}[E(k) - K(k)].$$

Here we notice from (1.11) that

$$\frac{2}{\sqrt{\pi\alpha}} \cdot \frac{d}{d\alpha} \left\{ E\left(\sqrt{\frac{\lambda}{\alpha}}\right) \right\} = \frac{1}{\sqrt{\pi} \cdot \alpha^{3/2}} E\left(\sqrt{\frac{\lambda}{\alpha}}\right) - \int_0^\infty e^{-\alpha x} L^{(0)}_{\frac{1}{2}}(\lambda x)\, x^{\frac{1}{2}}\, dx$$

(2.16)

$$= \frac{1}{\sqrt{\pi} \cdot \alpha^{3/2}} E\left(\sqrt{\frac{\lambda}{\alpha}}\right) - \frac{\sqrt{\pi}}{2\alpha^{3/2}} F\left(-\frac{1}{2}, \frac{3}{2}; 1; \frac{\lambda}{\alpha}\right).$$

Now we know that

$$(b-a)\, F(a,\, b;\, c;\, x) + aF(a+1) - bF(b+1) = 0.$$

In particular, we have

$$2F\left(-\frac{1}{2}, \frac{1}{2}; 1; \frac{\lambda}{\alpha}\right) - F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{\lambda}{\alpha}\right) - F\left(-\frac{1}{2}, \frac{3}{2}; 1; \frac{\lambda}{\alpha}\right) = 0$$

(2.17)

$$F\left(-\frac{1}{2}, \frac{3}{2}; 1; \frac{\lambda}{\alpha}\right) = 2F\left(-\frac{1}{2}, \frac{1}{2}; 1; \frac{\lambda}{\alpha}\right) - F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{\lambda}{\alpha}\right).$$

Thus it follows from (2.16) and (2.17) that

$$\frac{d}{d\alpha} \left\{ E\left(\sqrt{\frac{\lambda}{\alpha}}\right) \right\} = \frac{1}{2\alpha} \left[K\left(\sqrt{\frac{\lambda}{\alpha}}\right) - E\left(\sqrt{\frac{\lambda}{\alpha}}\right) \right]. \qquad (2.18)$$

Now using $\alpha = \frac{\lambda}{k^2}$, we easily derive from (2.18)

$$\frac{d}{dk} \{ E(k) \} = \frac{1}{k} [E(k) - K(k)].$$

REFERENCES

[1] A. Erdelyi, *Tables of Integral Transforms*, vol. 1, p 337 (Mc Graw-Hill Book Co, 1954).

[2] E. Pinney, *Laguerre functions in the mathematical foundations of the electromagnetic theory of the paraboloidal reflector*, «Jour. Math. Physics», vol. 25, 1946, pp. 49-79.

[3] A. Erdelyi, *Higher Transcendental Functions*, vol. 2, p. 318.

[4] L. Toscano, *Gli integrali ellittici completi di prima e seconda specie nel calcolo simbolico*, «Boll. Un. Mat. Ital.», Serie III, Anno V, 1950, pp. 236-238.