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On a formula of Kogbetliantz.

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Summary. - *A simple proof is given of* KOGBETLIANTZ'*s formula* (1) *below; a generalized version of this formula is given in* (5). *Also analogs of* (1), *for the* LAGUERRE *and* HERMITE *polynomials are obtained.*

1. KOGBETLIANTZ [1] has recently obtained the elegant formula

$$e^{x}f(x, y)+e^{y}f(y, x)=e^{x-y}-I_0(2\sqrt{xy}), \tag{1}$$

where

$$f(x, y)=\int_0^x e^{-t}I_0(2\sqrt{yt})dt \tag{2}$$

and $I_0(z)$ denotes the BESSEL function with imaginary argument. KOGBETLIANTZ has derived the formula by means of LAPLACE transformation. The formula can be obtained very simply as follows.

Since

$$I_0(2t)=\sum_{n=0}^{\infty}\frac{t^{2n}}{n!\,n!},$$

it follows from (2) that

$$f(x, y)=\sum_{n=0}^{\infty}\frac{y^n}{n!\,n!}\int_0^x e^{-t}t^n dt.$$

It is easily verified that

$$\int_0^x e^{-t}t^n dt=n!\left\{-e^{-x}\sum_{r=0}^{n}\frac{x^r}{r!}+1\right\},$$

so that

$$e^x f(x,\ y) = \sum_{n=0}^{\infty} \frac{y^n}{n!} \left\{ - \sum_{r=0}^{n} \frac{x^r}{r!} + e^x \right\}$$

$$= \sum_{n=0}^{\infty} \frac{y^n}{n!} \sum_{r=n+1}^{\infty} \frac{x^r}{r!} = \sum_{r=1}^{\infty} \frac{x^r}{r!} \sum_{n=0}^{r-1} \frac{y^n}{n!}.$$

Therefore

$$e^x f(x,\ y) + e^y f(y,\ x) = \sum_{r=1}^{\infty} \frac{x^r}{r!} \sum_{n=0}^{r-1} \frac{y^n}{n!} + \sum_{r=0}^{\infty} \frac{x^r}{r!} \sum_{n=r+1}^{\infty} \frac{y^n}{n!}$$

$$= \sum_{r=0}^{\infty} \frac{x^r}{r!} \sum_{n=0}^{\infty} \frac{y^n}{n!} - \sum_{r=0}^{\infty} \frac{(xy)^r}{r!\ r!}$$

$$= e^{x+y} - I_0(2\sqrt{xy}),$$

which evidently proves (1).

2. It may be of interest to mention that the functional equation

$$e^x f(x,\ y) + e^y f(y,\ x) = e^{x+y} - F(xy), \tag{3}$$

where

$$f(x,\ y) = \int_0^x e^{-t} F(yt)\,dt \tag{4}$$

and $F(z)$ is analytic about the origin, characterizes I_0. Indeed if we put

$$F(z) = \sum_{n=0}^{\infty} \frac{A_n z_n}{n!\ n!},$$

then as above

$$e^x f(x,\ y) = \sum_{n=0}^{\infty} \frac{A_n}{n!} y^n \sum_{r=n+1}^{\infty} \frac{x^r}{r!}.$$

It follows that

$$e^x f(x, y) + e^y f(y, x) = \sum_{n=0}^{\infty} \frac{A_n}{n!} y^n \sum_{r=n+1}^{\infty} \frac{x^r}{r!}$$

$$+ \sum_{r=0}^{\infty} \frac{A_r}{r!} x^r \sum_{n=r+1}^{\infty} \frac{y^n}{n!}$$

$$= \sum_{n=0}^{\infty} A_{\min(n,\ r)} \frac{x^r}{r!} \frac{y^n}{n!} - \sum_{n=0}^{\infty} \frac{A_n (xy)^n}{n!\ n!}.$$

Comparison with (3) gives $A_n = 1$, so that

$$F(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!\ n!} = I_0(2\sqrt{z}).$$

Hence the only function $F(z)$, analytic about the origin, that satisfies (3) and (4) is

$$F(z) = I_0(2\sqrt{z}).$$

3. KOGBETLIANTZ'S formula can be generalized as follows. Put

$$f_r(x, y) = \frac{\partial^r}{\partial y^r} f(x, y) = \int_0^x e^{-t} \frac{\partial^r}{\partial y^r} I_0(2\sqrt{yt})\, dt$$

$$= \int_0^x e^{-t} t^r (2\sqrt{yt})^{-r} I_r(2\sqrt{yt})\, dt.$$

Then we have

$$f_r(x, y) = \sum_{n=0}^{\infty} \frac{y^n}{n!\ (n+r)!} \int_0^x e^{-t} t^{n+r}\, dt,$$

so that

$$e^x f_r(x, y) = \sum_{n=0}^{\infty} \frac{y^n}{n!} \sum_{s=n+r+0}^{\infty} \frac{x^s}{s!}$$

$$= \sum_{n=0}^{\infty} \frac{y^n}{n!} \sum_{s=n+1}^{\infty} \frac{x^s}{s!} - \sum_{n=0}^{\infty} \frac{y^n}{n!} \sum_{s=n+1}^{n+r} \frac{x^s}{s!}.$$

It follows that

$$e^x f_r(x, y) + e^y f_r(y, x) = e^{x+y} - I_0(2\sqrt{xy})$$

$$- \sum_{s=1}^{r} \sum_{n=0}^{\infty} \frac{y^n x^{n+s}}{n!\ (n+s)!} - \sum_{s=1}^{r} \sum_{n=0}^{\infty} \frac{x^n y^{n+s}}{n!\ (n+s)!}.$$

Since

$$\sum_{n=0}^{\infty} \frac{y^n x^{n+s}}{n!\ (n+s)!} = x^s (2\sqrt{xy})^{-s} I_s(2\sqrt{xy}),$$

we get

$$e^x f_r(x, y) + e^y f_r(y, x) = e^{x+y} - I_0(2\sqrt{xy}) - \tag{5}$$

$$- \sum_{s=1}^{r} (x^s + y^s)(2\sqrt{xy})^{-s} I_s(2\sqrt{xy}).$$

For $r = 0$, (5) evidently reduces to (1).

4. For the Laguerre polynomial [2, p. 97]

$$L_n^{(\alpha)} = \sum_{r=0}^{n} \binom{n+\alpha}{n-s} \frac{(-x)^r}{r!}$$

we have

$$\int_0^x e^{-t} L_n^{(\alpha)}(yt)dt = \sum_{r=0}^{n} \binom{n+\alpha}{n-r} (-y)^r \Big\} - e^{-x} \sum_{s=0}^{r} \frac{x^{r-s}}{(r-s)!} + 1 \Big\}.$$

Since

$$\sum_{r=0}^{n} \binom{n+\alpha}{n-r} (-y)^r \sum_{s=0}^{r} \frac{x^{r-s}}{(r-s)!} = \sum_{s=0}^{n} (-y)^s \sum_{r=}^{n} \binom{n+\alpha}{n-r} \frac{(-xy)^{r-s}}{(r-s)!}$$

$$= \sum_{s=0}^{n} (-y)^s \sum_{r=0}^{n} \binom{n+\alpha}{n-s\frac{1}{2}r} \frac{(-xy)^r}{r!} = \sum_{s=0}^{n} (-y)^s L_{n-s}^{(\alpha+s)}(xy),$$

it follows that

$$e^x \int_0^x e^{-t} L_n^{(\alpha)}(xt)\,dt = - \sum_{s=0}^{n} (-y)^s L_{n-s}^{(\alpha+s)}(xy) \tag{6}$$

$$+ e^x \sum_{r=0}^{n} \binom{n+\alpha}{n-r} (-y)^r.$$

In particular for $\alpha = 0$, (6) reduces to

$$e^x \int_0^x e^{-t} L_n(yt)\,dt = \sum_{s=0}^{n} (-y)^s L_{n-s}^{(s)}(xy) + e^x(1-y)^n.$$

For the Hermite polynomial [2, p. 102]

$$H_n(x) = \sum_{2r \leq n} (-1)^r \frac{n!}{r!\,(n-2r)!} (2x)^{n-2r}$$

we get similarly

$$e^x \int_0^x e^{-t} H_n(yt)\,dt = - \sum_{s=0}^{n} \frac{n!}{s!} (2y)^{n-s} H_s(xy) \tag{7}$$

$$+ e^x \sum_{2r \leq n} (-1)^r \frac{n!}{r!} (2y)^{n-2r}.$$

REFERENCES

[1] E. G. KOGBETLIANTZ, *Sur une identitè remarquable concernant la fonction* I_0, Chiffres, 1 (1958), pp. 121-122.

[2] G. SZEGÖ, *Orthogonal polynomials,* New York, 1939.