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On the complete elliptic integral of the third kind

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Summary. - *It is shown that*

$$J(\alpha^2, k^2) = \frac{2}{\pi}\Pi(\alpha^2, k^2),$$

where $\Pi(\alpha^2, k^2)$ *is the complete elliptic integral of the third kind, satisfies the congruence* (8) *below.*

1. We recall that the complete elliptic integral of the first kind is given by

$$K = K(k^2) = \frac{\pi}{2} F\left(\frac{1}{2}, \frac{1}{2}; 1; k^2\right),$$

where F denotes the hypergeometric function. The writer has proved that if $p = 2m + 1$ is any odd prime then K satisfies the congruence [1]

$$\left\{ F\left(\frac{1}{2}, \frac{1}{2}; 1; k^2\right)\right\}^{p-1} \sum_{r=0}^{m} \binom{m}{r}^2 k^{2r} \equiv 1 \pmod{p}. \tag{1}$$

Similarly for the complete elliptic integral of the second kind

$$E = E(k^2) = \frac{\pi}{2} F\left(-\frac{1}{2}; \frac{1}{2}; 1; k^2\right)$$

we have [2]

$$\left\{ F\left(-\frac{1}{2}, \frac{1}{2}; 1; k^2\right)\right\}^{p-1} \sum_{r=0}^{m} \binom{m}{r}^2 k^{2r} \equiv \left\{ \sum_{r=0}^{m} \binom{m}{r}\binom{m+1}{r} k^{2r} \right\}^{p-1} \pmod{p}.$$

The complete elliptic integral of the third kind may be defined by

$$\Pi = \Pi(\alpha^2, k^2) = \int_0^{\frac{1}{2}\pi} \frac{d\Phi}{(1 - \alpha^2 \sin^2\Phi)\sqrt{1 - k^2 \sin^2\Phi}}.$$

Expanding and integrating term by term we get

$$\Pi(\alpha^2, k^2) = \frac{\pi}{2} J(\alpha^2, k^2),$$

where

$$J(\alpha^2, k^2) = \sum_{r,s=0}^{\infty} \frac{\left(\frac{1}{2}\right)_r \left(\frac{1}{2}\right)_{r+s}}{r!\,(r+s)!} k^{2r}\alpha^s. \tag{2}$$

Now

$$\frac{\left(\frac{1}{2}\right)_r}{r!} = \frac{1}{2^{2r}}\binom{2r}{r}.$$

Also if

$$r = r_0 + r_1 p + r_2 p^2 + \dots \qquad (0 \le r_j < p)$$

then

$$\binom{2r}{r} \equiv \binom{2r_0}{r}\binom{2r_1}{r_1}\binom{2r_2}{r_2}\dots \pmod{p}; \tag{3}$$

it follows in particular that $\binom{2r}{r} \equiv 0 \pmod{p}$ unless

$$0 \le r_j < m \qquad (j = 0, 1, 2, \dots).$$

Since by (2)

$$J(\alpha^2, \alpha^2 k^2) = \sum_{t=0}^{\infty} \sum_{r=0}^{t} \frac{\left(\frac{1}{2}\right)_r \left(\frac{1}{2}\right)_t}{r!\ t!} k^{2r}\alpha^{2t},$$

it follows from (3) that

$$J(\alpha^2, \alpha^2 k^2) \equiv \sum_{t=0}^{\infty} \binom{2t_0}{t_0}\binom{2t_1}{t_1}\dots \sum_{r=0}^{t} \binom{2r_0}{r_0}\binom{2r_1}{r_1}\dots \frac{k^{2r}}{2^{2r}}\frac{\alpha^{2t}}{2^{2t}} \pmod{p} \dots,$$

where

$$t = t_0 + t_1 p + t_2 p^2 + \dots \qquad (0 \le t_j < p).$$

We now define

$$J_n(\alpha^2, \alpha^2 k^2) = \sum_{t=0}^{p^{n+1}-1} \binom{2t_0}{t_0}\binom{2t_1}{t_1} \cdots \sum_{r=0}^{t} \binom{2r_0}{r_0}\binom{2r_1}{r_1} \cdots \frac{k^{2r}}{2^{2r}}\frac{\alpha^{2t}}{2^{2t}}.$$

Since

$$\frac{1}{2^{2r}}\binom{2r}{r} \equiv (-1)^r \binom{m}{r} \pmod{p} \qquad (0 \leq r < p),$$

it follows that

$$J_n(\alpha^2, \alpha^2 k^2) \equiv \sum_{t=0}^{p^{n+1}-1} (-1)^t \binom{m}{t_0}\binom{m}{t_1} \cdots \sum_{r=0}^{t} (-1)^r \binom{m}{r_0}\binom{m}{r_1} \cdots k^{2r}\alpha^{2t} \pmod{p}.$$

Now if

$$r = r_0 + r_1 p + \dots + r_n p^n, \qquad t = t_0 + t_1 p + \dots + t_n p^n$$

and $r_n < t_n$ then $r_0, \dots, r_{n-1}$ can be chosen arbitrarily; if $r_n = t_n$ while $r_{n-1} < t_{n-1}$, then $r_0, \dots, r_{n-2}$ are arbitrary; if $r_n = t_n$, $r_{n-1} = t_{n-1}$ while $r_{n-2} < t_{n-2}$, then $r_0, \dots, r_{n-3}$ are arbitrary, and so on. It follows that

$$J_n(\alpha^2, \alpha^2 k^2) \equiv \sum_{j=0}^{n} U_j + U, \tag{4}$$

where

$$U = \sum_{t_0, \dots, t_n = 0}^{m} \binom{m}{t_0}^2 \cdots \binom{m}{t_0}^2 (\alpha^2 k^2)^{t_0 + \dots + t_n p^n},$$

$$U_j = \sum_{t_0, \dots, t_j} (-1)^{t_0 + \dots + t_j} \binom{m}{t_0} \cdots \binom{m}{t_j} \alpha^{2(t_0 + \dots + t_j p^j)}$$

$$\cdot \sum_{r_0, \dots, r_j} (-1)^{r_0 + \dots + r_j} \binom{m}{r_0} \cdots \binom{m}{r_j} k^{2(r_0 + \dots + r_j p^j)}$$

$$\cdot \sum_{t_{j+1}, \dots, t_n} \binom{m}{t_{j+1}}^2 \cdots \binom{m}{t_n}^2 (\alpha^2 k^2)^{t_{j+1} p^{j+1} + \dots + t_n p^n},$$

where each t_i runs from 0 to m; also $r_0, \dots, r_{j-1}$ run from 0 to m but r_j runs from 0 to $t_j - 1$.

It for brevity we put

$$S(\alpha^2) = \sum_{r=0}^{m} \binom{m}{r}^2 \alpha^{2r}, \tag{5}$$

$$T(\alpha^2, k^2) = \sum_{t=0}^{m} (-1)^t \binom{m}{t} \sum_{r=0}^{t-1} (-1)^r \binom{m}{r} k^{2r}\alpha^{2t}, \tag{6}$$

it is clear that

$$U = \prod_{r=0}^{n} S((\alpha^2 k^2)^{p^r}) \equiv (S(\alpha^2 k^2))^{1+p+\cdots+p^n} \qquad (\mathrm{mod}\ p),$$

$$U_j \equiv \{(1-\alpha^2)^m (1-k^2)^m\}^{1+p+\cdots+p^{j-1}} \cdot T^{p^j}(\alpha^2, k^2)\,(S(\alpha^2 k^2))^{p^{j+1}+\cdots+p^n}.$$

Hence (4) becomes

$$J_n(\alpha^2, \alpha^2 k^1) \equiv (S(\alpha^2 k^2))^{1+p+\cdots+p^n}$$

$$+ \sum_{j=0}^{n} \{((1-\alpha^r)(1-k^2))\}^{(p^j-1)} T^{p^j}(\alpha^2, k^2) \cdot (S(\alpha^2, k^2))^{p^{j+1}+\cdots+p^n}.$$

It follows that

$$((1-\alpha^2)(1-k))^{\frac{1}{2}} \{J_n(\alpha^2, \alpha^2 k^2) - (S(\alpha^2 k^2))^{1+\cdots+p^n}$$

$$\equiv \sum_{j=0}^{n} ((1-\alpha^2)(1-k^2))^{\frac{1}{2}p^j}\; T^{p^j}(\alpha^2, k^2)(S(\alpha^2 k^2))^{p^{j+1}+\cdots+p^n}.$$

If we denote the left member by W_n, then it is evident that

$$W_n - W_{n-1}^p \equiv ((1-\alpha^2)(1-k^2))^{\frac{1}{2}} T(\alpha^2, k^2)(S(\alpha^2 k^2))^{p+\cdots+p^n}.$$

Thus

$$J_n(\alpha^2, \alpha^2 k^2) - (1-\alpha^2)^m (1-k^2)^m J_{n-1}^p(\alpha^2, \alpha^2 k^2)$$

$$(S(\alpha^2 k^2))^{1+\cdots+p^n} - (1-\alpha^2)^m (1-k^2)^m (S(\alpha^2 k^2))^{p+\cdots+p^n}$$

$$+\ T(\alpha^2, k^2)(S(\alpha^2 k^2))^{p+\cdots+p^n},$$

which yields

$$(S(\alpha^2k^2))^{-(p+\dots+p^n)}\{J_n(\alpha^2,\ \alpha^2k^2)-(1-\alpha^2)^m(1-k^2)^m J^{p_n}_{n-1}(\alpha^2,\ \alpha^2k^2)\}$$

$$\equiv S(\alpha^2k^2)-(1-\alpha^2)^m(1-k^2)^m+T(\alpha^2,\ k^2).$$

We now let $n\to\infty$. Since (1) implies

$$F\left(\frac{1}{2},\frac{1}{2};\ 1;\ k^2\right)\left\{\sum_{r=0}^{m}\binom{m}{r}^2k^{2r}\right\}^{1/(p-1)}\equiv 1,$$

it follows that

$$(7)\qquad F^p\left(\frac{1}{2}\cdot\frac{1}{2};\ 1;\ \alpha_1k^2\right)\{J(\alpha^2,\alpha^2k^2)-(1-\alpha^2)^m(1-k^2)^m J^p(\alpha^2,\alpha^2k^2)\}$$

$$\equiv S(\alpha^2k^2)-(1-\alpha^2)^m(1\ k^2)^m+T(\alpha^2k^2)\qquad(\mathrm{mod}\ p)$$

The right member can be simplified further. By (5) and (6) we have

$$(1-\alpha^2)^m(1-k^2)^m-T(\alpha^2,k^2)$$

$$=\sum_{r,s=0}^{m}(-1)^{r+s}\binom{m}{r}\binom{m}{s}\alpha^{2r}k^{2s}-\sum_{r=0}^{m}\sum_{s=0}^{r-1}(-1)^{r+s}\binom{m}{r}\binom{m}{s}\alpha^{2r}k^{2s}$$

$$=\sum_{r=0}^{m}\sum_{s=r}^{m}(-1)^{r+s}\binom{m}{r}\binom{m}{s}\alpha^{2r}k^{2s}$$

$$=\sum_{s=0}^{r}\sum_{r=0}^{s}(-1)^{r+s}\binom{m}{r}\binom{m}{s}k^{2s}\alpha^{2r}$$

$$=T(k^2,\alpha^2)+S(\alpha^2k^2),$$

so that (7) becomes

$$(8)\qquad F^p\left(\frac{1}{2},\frac{1}{2};\ 1,\alpha^2k^2\right)\{J(\alpha^2,\alpha\ k^2)-(1-\alpha^2)^m(1-k^2)^m J^p(\alpha^2,\alpha^2k^2)\}$$

$$\equiv -\ T(k^2,\alpha^2)\qquad(\mathrm{mod}\ p)$$

2. We have incidentally proved that

(9) $$T(\alpha^2, k^2) + T(\alpha^2, k^2) = (1-\alpha^2)^m (1-k^2)^m - S(\alpha^2 k^2).$$

If we put

$$J_n'(\alpha^2, \alpha^2 k^2) = \sum_{t=0}^{p^{n+1}-1} \sum_{r=0}^{t} \frac{\left(\frac{1}{2}\right)_r \left(\frac{1}{2}\right)_t}{r!\,t!} k^{2r}\alpha^{2t},$$

it follows that

$$J_n'(\alpha^2, \alpha^2 k^2) + J_n'(k^2, \alpha^2 k^2)$$

$$= \sum_{r,s=0}^{p^{n+1}-1} \frac{\left(\frac{1}{2}\right)_r \left(\frac{1}{2}\right)_s}{r!\,s!} k^{2r}\alpha^{2s} + \sum_{r=0}^{p^{n+1}-1} \left(\frac{\left(\frac{1}{2}\right)_r}{r!}\right)^2 (k\alpha)^{2r}$$

Therefore we get

$$J(\alpha^2, \alpha^2 k^2) + J(k^2, \alpha^2 k^2)$$

$$= (1-\alpha^2)^{-\frac{1}{2}} (1-k^2)^{-\frac{1}{2}} + F\left(\frac{1}{2}, \frac{1}{2}; 1; \alpha^2 k^2\right),$$

or what is the same thing

(10) $$\Pi(\alpha^2, \alpha^2 k^2) + \Pi(k^2, \alpha^2 k^2)$$

$$= \frac{\pi}{2}(1-\alpha^2)^{-\frac{1}{2}} (1-k^2)^{-\frac{1}{2}} + K(\alpha^2 k^2).$$

The special case

$$\Pi(k, k^2) = \frac{\pi}{4}(1-k)^{-1} + \frac{1}{2} K(k^2)$$

is familiar.

The formula (10) is due to Legendre.

REFERENCES

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[2] L. Carlitz, *An arithmetic property of the complete elliptic integral of the second kind*, to appear.