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Edison Farah

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A New Proof of Zorn's Theorem.

Nota di EDISON FARAH (a S. Paulo)

Summary. - *The author gives, in this note, a proof Zorn's Theorem which is obtained as a direct consequence of the Well-ordering Theorem.*

In this note we present a new proof of *Zorn's Theorem* (concerning the existence of a maximal element in an ordered inductive set), based on the fact that every set may be well ordered (ZERMELO's Theorem). Although the literature already contains several proofs of the ZORN Theorem (the latest known to as being that of [2] (1), p. 45), it seemed to us worthwhile to publish the present one, since it is a simple application of the ZERMELO Theorem about the well ordering of an arbitrary set.

The notations and terminology used here are those of [1] (or [2]). In the definition of an ordered inductive set we do not require the existence of least upper bound for totally ordered parts, but only the existence of an upper bound. More precisely : an ordered set E is said to be *inductive* if every totally ordered subset of E is bounded above (Cf., for example, [3], p. 42, formulation (AC2) of the Axiom of Choice). We shall prove :

ZORN's THEOREM. - *Every inductive ordered set has a maximal element.*

PROOF. - Let E be a set inductive relative to the order $x \leq y$, which we shall call ω; According to ZERMELO's Theorem, E may be well ordered by an order relation $x \preccurlyeq y$, which we shall denote by ω'. Consider the class $\mathfrak{F}$ of the subsets X of E which satisfy the following conditions :

(*a*) X is ω-totally ordered and the ω'-first element of E belongs to X;

(*b*) if u belongs to X an v is any element of E, strictly ω-comparable with all the elements of X which are strictly ω'-inferior to u, then $u \preccurlyeq v$.

(Note that the unitary set formed by the ω'-first element of E satisfies (*a*) and (*b*)).

Then, let E^* be the union of all the sets of $\mathfrak{F}$; we will show that E^* is ω-totally ordered. In fact, if this were not the case, letting x_0 be the ω'-first element of E not ω-comparable to some element of E^+, and y_0 the ω'-first element of E not

(1) The numbers in brackets refer to the bibliography at the end of this note.

ω-comparable to x_0, we would evidently have $x_0 \prec y_0$. On the other hand, taking $Y \in \mathfrak{F}$ such that $y_0 \in Y$, we would certainly have $x_0 \notin X$, whence, taking into consideration that y_0 is a minimum, and applying condition (*b*) to Y (choosing for u and v, y_0 and x_0 respectively), one would have $y_0 \preccurlyeq x_0$, which is absurd. Consequently, E^* is ω-totally ordered and, therefore, admits an ω-majorant m^* in E. Let us prove that m^* is an ω-maximal element of E. For, if this is not so, letting m be the least (in the order ω') element of E strictly ω-superior to m^*, the set $E^* \cup \{m\}$ will belong to $\mathfrak{F}$. In fact, first it is clear that $E^* \cup \{m\}$ satisfies condition (*a*). On the other hand, take elements u in $E^* \cup \{m\}$ and v in E, as in condition (*b*). Supposing first that $u \neq m$, then u will belong to a certain subset of E which satisfies (*b*), and hence $u \preccurlyeq v$. In particular (letting m play the role of v), we would have $x \prec m$ for all x of E^*, whence, if $u = m$, considering the minimum condition imposed on m, it follows that $m \preccurlyeq v$. Summarizing, $E^* \cup \{m\}$ also satisfies condition (*b*) and, consequently, belongs to $\mathfrak{F}$, which is impossible, since E^* is the union of all the sets of $\mathfrak{F}$ and $m \notin E^*$. Therefore, m^* is a maximal element of E in the order ω.

BIBLIOGRAPY

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[2] N. BOURBAKI, *Théorie des ensembles*, Chap. III, 1956.
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