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Certain identities and inequalities concerning Hermite polynomials.

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Certain identities and inequalities concerning Hermite polynomials.

Nota di S. K. Chatterjea (a Calcutta)

Summary. - *For the* Hermite *polynomials* $H_n(x)$ *we define the following function of the real variable* x

$$\Delta_{n,\,h,\,k\,;\,x}(H) = H_{n+h}(x)H_{n+k}(x) - H_n(x)H_{n+h+k}(x),$$

where n, h, k *are integers such that* $n \geq 0$, $k \geq h \geq 1$. *The results we wish to prove are the following:*

(1) $$\Delta_{n-1,\,1,\,1\,;\,x}(H) \geq 2^n(n-1)\,!$$

(2) $$\Delta_{n-1,\,1,\,1\,;\,x}(H) \geq 2H^2_{n-1}(x)$$

(3) $$\delta_{n,\,1,\,2\,;\,x}(H) \equiv (n+2)H_{n+1}(x)H_{n+2}(x) - (n+1)H_n(x)H_{n+3}(x) = \\ = x[\,\{(H'_{n+1}(x))^2 - H_{n+1}(x)H''_{n+1}(x)\} + 2H^2_{n+1}(x)]$$

(4) $$\delta_{n,\,1,\,2\,;\,x}(H) \quad \left.\begin{matrix} <0, & x<0 \\ =0, & x=0 \\ >0, & x>0 \end{matrix}\right\} ; \qquad n \geq 0$$

(5) $$\sigma_{n,\,1,\,2\,;\,x}(H) \equiv (n+3)H_{n+1}(x)H_{n+2}(x) - (n+1)H_n(x)H_{n+3}(x) = \\ = \frac{1}{2}\Delta_{n+1,\,1,\,2\,;\,x}(H)$$

(6) $$\sigma_{n,\,1,\,2\,;\,x}(H) \quad \left.\begin{matrix} <0, & x<0 \\ =0, & x=0 \\ >0, & x>0 \end{matrix}\right\} ; \qquad n \geq 0.$$

(7) $$\Delta_{n,\,1,\,2\,;\,x}(H) = 4\left[2^{n+1}\cdot n\,!\,x + \sum_{r=0}^{n-1} 2^r \binom{n}{r} r\,!\,H_{n-r}(x\,H_{n-r+1}(x)\right]$$

(8) $$\Delta_{n,\,1,\,1\,;\,x}(H) = (H'_n(x))^2 - H_n(x)H''_n(x) + 2H^2_n(x)$$

(9) $$\Delta_{n,\,1,\,2\,;\,x}\left(\frac{d^k}{dx^k}H\right) \quad \left.\begin{matrix} <0, & x<0 \\ =0, & x=0 \\ >0, & x>0 \end{matrix}\right\} ; \qquad 1 \leq k \leq n.$$

(10) $$\Delta_{n,\,1,\,2\,;\,x}(H) = 2n\Delta_{n-1,\,1,\,2\,;\,x}(H) + 4H_n(x)H_{n+1}(x) = \\ = 8xH^2_n(x) + 4n(n-1)\Delta_{n-2,\,1,\,2\,;\,x}(H).$$

1. If we define the HERMITE polynomials by the recurrence relation

(1.1) $$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x), \qquad H_0(x) = 1, \qquad H_1(x) = 2x.$$

We know that they satisfy the relations

(1.2) $$H'_n(x) = 2nH_{n-1}(x)$$

(1.3) $$H'_n(x) = 2xH_n(x) - H_{n+1}(x)$$

(1.4) $$H''_n(x) - 2xH'_n(x) + 2nH_n(x) = 0.$$

Now it is well-known that [1]

(1.5) $$\Delta_{n-1,1,1:x}(H) = 2^n(n-1)!\left[1 + \sum_{r=1}^{n-1} \frac{H^2_{n-r}(x)}{2^{n-r}(n-r)!}\right]$$

∴ it follows that

(1.6) $$\Delta_{n-1,1,1;x}(H) \geq 2^n(n-1)!$$

Again from (1.5) we have $\Delta_{n-2,1,1;x}(H) > 0$. Also we know that $\Delta_{n-1,1,1;x}(H) - 2H^2_{n-1}(x) = 2(n-1)\Delta_{n-2,1,1;x}(H) \geq 0$, $(n \geq 1)$ ∴ we have

(1.7) $$\Delta_{n-1,1,1;x}(H) \geq 2H^2_{n-1}(x).$$

Next, TOSCANO [2] proved that

(1.8) $$\Delta'_{n-1,1,1;x}(H) = 2(n-1)\Delta_{n-2,1,2,x}(H),$$

and

(1.9) $$\Delta'_{n-1,1,1;x}(H) = (16x)\sum_{i=0}^{\left[\frac{n-2}{2}\right]} 2^i(2i+1)!\binom{n-1}{2i+1} H^2_{n-2-2i}(x).$$

Thus it follows from (1.8) and (1.9) that

(1.10) $$(n-1)\Delta_{n-2,1,2;x}(H) = (8x)\sum_{i=0}^{\left[\frac{n-2}{2}\right]} 2^i(2i+1)!\binom{n-1}{2i+1} H^2_{n-2-2i}(x).$$

Hence

(1.11) $$\Delta_{n,1,2;x}(H) \quad \left.\begin{array}{ll} < 0, & x < 0 \\ = 0, & x = 0 \\ > 0, & x > 0 \end{array}\right\}; \qquad n \geq 0$$

which was indicated by DANESE [3].

2. Now we shall prove that

$$(2.1)\qquad \delta_{n,\,1,\,2\,;\,x}(H) \equiv (n+2)H_{n+1}(x)H_{n+2}(x) - (n+1)H_n(x)H_{n+3} \lesseqqgtr 0,$$

according as $-\infty < x < 0$, or $x = 0$ or $0 < x < \infty$; $(n \geq 0)$.

For,

$$2(n+1)\Delta_{n,\,1,\,2\,,\,x}(H)$$
$$= 2(n+1)H_{n+1}\cdot(2xH_{n+1} - H'_{n+1}) - H'_{n+1}\cdot(2xH_{n+2} - H'_{n+2})$$
$$= 2(n+1)H_{n+1}\cdot(2xH_{n+1} - H'_{n+1})$$
$$- H_{n+1}\{2x(2xH_{n+1} - H'_{n+1}) - 2(n+2)H_{n+1}\}$$

$$\therefore\ (n+1)\Delta_{n,\,1,\,2\,.\,x}(H)$$
$$=2(n+1)xH^2_{n+1}(x)+H_{n+1}(x)H'_{n+1}(x)-2x^2H_{n+1}(x)H'_{n+1}(x)+\cdot x(H'_{n+1}(x))^2$$
$$= x[(H'_{n+1}(x))^2 - H_{n+1}(x)H''_{n+1}(x)] + H_{n+1}(x)H'_{n+1}(x).$$

Thus we get

$$(2.2)\qquad (n+1)\Delta_{n,\,1,\,2\,,\,x}(H) = x[\{(H'_{n+1}(x))^2 - H_{n+1}(x)H''_{n+1}(x)\} + 2H^2_{n+1}(x)] - H_{n+1}(x)\cdot H_{n+2}(x)$$

$$(2.3)\qquad \therefore\ (n+2)H_{n+1}(x)H_{n+2}(x) - (n+1)H_n(x)H_{n+3}(x) = x[\{(H'_{n+1}(x))^2 - H_{n+1}(x)H''_{n+1}(x)\} + 2H^2_{n+1}(x)].$$

Since the roots of $H_n(x)$ are real and simple, we have [4]

$$(2.4)\qquad (H'_n(x))^2 - H_n(x)H''_n(x) > 0, \qquad -\infty < x < \infty.$$

Hence (2.1) follows from (2.3) and (2.4).

Furthermore we observe that

$$(2.5)\qquad \Delta_{n,\,1,\,2\,;\,x}(H) = H_{n+2}(2xH_n - 2nH_{n-1}) - H_n\{2xH_{n+2} - 2(n+2)H_{n+1}\} = 2[(n+2)H_nH_{n+1} - nH_{n-1}H_{n+2}]$$

$\therefore$ from (1.11) and (2.5) we get

$$(2.6)\qquad \sigma_{n,\,1,\,2\,;\,x}(H) \equiv (n+3)H_{n+1}(x)H_{n+2}(x) - (n+1)H_n(x)H_{n+3}(x) \lesseqqgtr 0,$$

according as $-\infty < x < 0$, or $x = 0$, or $0 < x < \infty$; $(n \geq 0)$.

Combining (1.11), (2.1) and (2.6) we get the result:

$$(2.7)\qquad (n+r)H_{n+1}(x)H_{n+2}(x) - (n+1)H_n(x)H_{n+3}(x) \lesseqqgtr 0,$$

according as $-\infty < x < 0$, or $x = 0$, or $0 < x < \infty$, when $r = 1, 2, 3$; $n \geq 0$.

$4H_1(x)H_2(x) - H_0(x)H_3(x) = 4x(6x^2 - 1)$ shows that r cannot be replaced by 4 in (2.7).

3. In this section we derive some identities whereby the inequality (2.1) and TURAN'S inequality for HERMITE polynomials are rendered intutive. To this end, we notice from (2.5) that

$$\Delta_{n,1,2;x}(H) = 2n\Delta_{n-1,1,2,x}(H) + 4H_n(x)H_{n+1}(x), \tag{3.1}$$

which immediately leads to the relation

$$\Delta_{n,1,2,x}(H) = 4\left[2^{n+1}\cdot n!\,x + \sum_{r=0}^{n-1} 2^r \binom{n}{r} r!\, H_{n-r}(x)H_{n-r+1}(x)\right] \tag{3.2}$$

we again have

$$\begin{aligned}\Delta_{n,1,2;x}(H) &= \begin{vmatrix} H_{n+1}(x) & H_n(x) \\ 2xH_{n+2}(x)-2(n+2)H_{n+1}(x) & 2xH_{n+1}(x)-2(n+1)H_n(x)\end{vmatrix} \\ &= 2x\begin{vmatrix} H_{n+1}(x) & H_n(x) \\ H_{n+2}(x) & H_{n+1}(x)\end{vmatrix} + 2\begin{vmatrix} 2H_{n+1}(x) & H_n(x) \\ H_{n+1}(x) & H_n(x)\end{vmatrix}, \\ &= 2x\Delta_{n,1,1;x}(H) + 2H_n(x)H_{n+1}(x).\end{aligned} \tag{3.3}$$

Now from (3.1) and (3.3) we get

$$x\Delta_{n,1,1;x}(H) = n\Delta_{n-1,1,2;x}(H) + H_n(x)H_{n+1}(x)$$

$$\therefore\ (n+1)H_n(x)H_{n+1}(x) - nH_{n-1}(x)H_{n+2}(x) = x\Delta_{n,1,1;x}(H). \tag{3.4}$$

Herewith the inequality (2.1) or TURAN'S inequality can be easily proved. (3.4) was also obtained by DANESE in [3].

Next comparing (2.3) and (3.4) we derive

$$H^2_{n+1}(x) - H_n(x)H_{n+2}(x) = (H'_n(x))^2 - H_n(x)H''_n(x) + 2H^2_n(x). \tag{3.5}$$

Herewith TURAN'S inequality for HERMITE polynomials is easily established with the help of (2.4).

4. Now we shall study the behaviour of

$$\Delta_{n,1.2;x}\left(\frac{d^k}{dx^k}H\right) \equiv \frac{d^k}{dx^k}(H_{n+1}(x))\cdot\frac{d^k}{dx^k}(H_{n+2}(x)) - \frac{d^k}{dx^k}(H_n(x))\cdot \cdot\frac{d^k}{dx^k}(H_{n+3}(x)), \qquad 1 \leq k \leq n.$$

The derivatives of the HERMITE polynomials satisfy the recurrence formula

$$(4.1)\qquad \frac{d^k}{dx^k} H_n(x) = 2^k \frac{n!}{(n-k)!} H_{n-k}(x), \qquad k \leq n.$$

Now

$$\Delta_{n,1,2;x}\left(\frac{d^k}{dx^k} H\right) = 2^{2k} \frac{n!\,(n+2)!}{(n-k+1)!\,(n-k+3)!} \cdot$$

$$(4.2)\qquad \cdot [(n+1)(n-k+3)H_{n-k+1}H_{n-k+2} - (n+3)(n-k+1)\cdot H_{n-k}H_{n-k+3}] = \lambda(n-k+1)(n-k+3)\Delta_{n-k,1,2;x}(H) + \lambda k \cdot \sigma_{n-k,1,2;x}(H),$$

where

$$\lambda = 2^{2k} \frac{n!\,(n+2)!}{(n-k+1)!\,(n-k+3)!}.$$

Thus it follows from (1.11), (2.6) and (4.2) that

$$(4.3)\qquad \Delta_{n,1,2;x}\left(\frac{d^k}{dx^k} H\right) \begin{matrix} <0, & x<0 \\ =0, & x=0 \\ >0, & x>0 \end{matrix} \Bigg\}; \qquad 1 \leq k \leq n.$$

5. The last section deals with the combination $\Delta_{n,1,2;x}(H)$, furnishing a simple proof of (1.10). From (2.5) we observe that

$$(5.1)\qquad \begin{aligned} \Delta_{n,1,2,x}(H) &= 2n\Delta_{n-1,1,2;x}(H) + 4H_n(x)H_{n+1}(x) \\ &= 2n[2(n-1)\Delta_{n-2,1,2;x}(H) + 4H_{n-1}(x)H_n(x) \\ &\quad + 4H_n(x)\cdot[2xH_n(x) - 2nH_{n-1}(x)] \\ &= 8xH^2_n(x) + 4n(n-1)\Delta_{n-2,1,2;x}(H). \end{aligned}$$

∴ by repeated application of (5.1) we get (1.10).

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