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Some integral equations satisfied by theta functions

Nota di LEONARD CARLITZ (ad Amherst, U. S. A.)

Summary. - *Making use of the orthogonality of certain sequences of poly nomials, it is shown that the theta functions satisfy certain integral equations.*

Put

$$H_n(x) = \sum_{r=0}^{n} \begin{bmatrix} n \\ r \end{bmatrix} x^r, \quad G_n(x) = \sum_{r=0}^{n} \begin{bmatrix} n \\ r \end{bmatrix} q^{r(r-n)} x^r,$$

where

$$\begin{bmatrix} n \\ r \end{bmatrix} = \frac{(q)_n}{(q)_r\,(q)_{n-r}}, \quad \begin{bmatrix} n \\ 0 \end{bmatrix} = \begin{bmatrix} n \\ n \end{bmatrix} = 1, \quad (q)_r = (1-q)(1-q^2)\dots(1-q^r), \quad (q)_0 = 1.$$

Szegö [2] showed that

$$\frac{1}{2\pi}\int_0^{2\pi} H_m(-q^{-\frac{1}{2}}e^{i\varphi})\, H_n(-q^{-\frac{1}{2}}e^{-i\varphi})\, f(\varphi)\,d\varphi = q^{-n}(q)_n\delta_{mn}, \tag{1}$$

where

$$f(\varphi) = \sum_{n=-\infty}^{\infty} q^{\frac{1}{2}n^2} e^{ni\varphi} \qquad (|q| < 1). \tag{2}$$

Wigert [3] proved that

$$\int_0^{\infty} G_m\left(-q^{m+\frac{1}{2}}x\right) G_n\left(-q^{n+\frac{1}{2}}x\right) p(x)\,dx = q^{-n-\frac{1}{2}}(q)_n\delta_{mn}, \tag{3}$$

where

$$p(x) = k\pi^{-\frac{1}{2}} \exp\left(-k^2 \log^2 x\right) \tag{4}$$

and

$$2k^2 = -\frac{1}{\log q} \qquad (0 < q < 1). \tag{5}$$

In a recent paper [1], the writer has given simple proofs of (1) and (3) as well as some related formulas; in particular the new orthogonality relation

$$\int_0^\infty G_m(-q^{\frac{1}{2}}x)G_n(-q^{\frac{1}{2}}x^{-1})x^{-1}p(x)dx = (-1)^n q^{-\frac{1}{2}n(n-1)}(q)_n\delta_{mn} \tag{6}$$

may be cited. It was also noted that (1) implies the following relation

$$\frac{1}{2\pi}\int_0^{2\pi} \prod_{n=0}^{\infty} \frac{(1+q^{n+\frac{1}{2}}e^{i\varphi})(1+q^{n+\frac{1}{2}}e^{-i\varphi})}{(1+q^{n+\frac{1}{2}}e^{i\varphi}t)(1+q^{n+\frac{1}{2}}e^{-i\varphi}z)}\,d\varphi \tag{7}$$

$$= \prod_{n=1}^{\infty} \frac{(1-q^n t)(1-q^n z)}{(1-q^n)(1-q^n tz)},$$

which holds for

$$|q^{\frac{1}{2}}t| < 1, \qquad |q^{\frac{1}{2}}z| < 1, \tag{8}$$

Similary (6) implies

$$\int_0^\infty \prod_{n=1}^{\infty}(1+q^n xt)(1+q^n x^{-1}z)x^{-1}p(x)dx = \prod_{n=1}^{\infty}\frac{1-q^n tz}{1}$$

which also holds when (8) is satisfied. (Note that this formula ((6.2) of [1]) is not quite correct as it stands).

It is not difficult to prove (7) and (9) directly. Thus to prove (9), we take

$$\int_0^\infty \prod_{n=1}^{\infty}(1+q^n xt)\cdot\prod_{n=1}^{\infty}(1+q^n x^{-1}z)\cdot x^{-1}p(x)dx$$

$$\sum_{r,s=0}^{\infty}\frac{q^{\frac{1}{2}r(r+1)}}{(q)_r}\,\frac{q^{-\frac{1}{2}s(s+1)}}{(q)_s}\,t^r z^s \int_0^\infty x^{r-s-1}p(x)dx.$$

But by formula (4.1) of [1]

$$\int_0^\infty x^n p(x)dx = q^{-\frac{1}{2}(n+1)^2}$$

for all n. Hence we get

$$\sum_{r,\,s=0}^{\infty} \frac{q^{\frac{1}{2}r(r+1)}}{(q)_r} \frac{q^{\frac{1}{2}s(s+1)}}{(q)_s} q^{-\frac{1}{2}(r-s)^2} t^r z^s$$

$$= \sum_{r,\,s=0}^{\infty} \frac{q^{\frac{1}{2}(r+s)} q^{rs}}{(q)_r (q)_s} t^r z^s = \sum_{r=0}^{\infty} \frac{q^{\frac{1}{2}r} t^r}{(q)_r} \sum_{s=0}^{\infty} \frac{q^{\left(\frac{1}{2}+r\right)s} z^s}{(q)_s}$$

$$= \sum_{r=0}^{\infty} \frac{q^{\frac{1}{2}r} t^r}{(q)_r} \prod_{n=0}^{\infty} (1 - q^{\frac{1}{2}+r+n} z)^{-1} = \prod_{n=0}^{\infty} (1 - q^{n+\frac{1}{2}} z)^{-1} \sum_{r=0}^{\infty} q^{\frac{1}{2}r} t^r \frac{(q^{\frac{1}{2}} z)_r}{(q)_r}$$

$$= \prod_{n=1}^{\infty} \frac{1 - q^n t z}{(1 - q^{n-\frac{1}{2}} t)(1 - q^{n-\frac{1}{2}} z)},$$

where we have used the notation

$$(a)_r = (1-a)(1-qa) \dots (1 - q^{r-1} a), \qquad (a)_0 = 1.$$

This evidently completes the proof of (9). The interchange of integration and summation is justified by absolute convergence. Moreover it is clear from the above that (9) holds provided (8) is satisfied;

2. Returning to (7), we take $z = t^{-1}$ and replace q by q^2. Thus (7) becomes

$$\frac{1}{2\pi} \int_0^{\infty} \prod_{n=1}^{\infty} \frac{(1 - q^{2n-1} e^{i\varphi})(1 - q^{2n-1} e^{-i\varphi})}{(1 - q^{2n-1} e^{i\varphi} t)(1 - q^{2n-1} e^{-i\varphi} t^{-1})} d\varphi \tag{10}$$

$$= \prod_{n=1}^{\infty} \frac{(1 - q^{2n} t)(1 - q^{2n} t^{-1})}{(1 - q^{2n})^2}.$$

where now, in view of (8)

$$|q|^{-\frac{1}{2}} < |t| < |q|^{\frac{1}{2}}. \tag{11}$$

In particular we may take $t = e^{i\omega}$, where ω is real.

Next if we recall that

$$\theta_3(v) = \theta_3(v, q) = \sum_{n=-\infty}^{\infty} q^{n^2} e^{2n\pi v i}$$

$$= \prod_{n=1}^{\infty} (1 - q^{2n})(1 + q^{2n-1} e^{2\pi v i})(1 + q^{2n-1} e^{-2\pi v i}),$$

$$\theta_1(v)_v\,\theta_1(v,\ q) = \frac{1}{i}\sum_{n=-\infty}^{\infty}(-1)^n q^{(2n+1)^2} e^{(2n+1)\pi vi}$$

$$= 2q^{\frac{1}{4}}\sin\pi_v \cdot \prod_{n=1}^{\infty}(1-q^{2n})(1-q^n e^{2\pi vi})(1-q^n e^{-2\pi vi}),$$

then it is clear that (10) implies

$$\int_0^1 \frac{\theta_3(\varphi,\ q)}{\theta_3(\varphi+\omega,\ q)}\,d\varphi = \frac{1}{2}q^{-\frac{1}{4}}\frac{\theta_1(\omega,\ q)}{\sin\pi\omega}. \tag{12}$$

Note that the left member of (12) can also be written in the form

$$\int_0^1 \frac{\theta_3(\varphi+\omega,\ q)}{\theta_3(\varphi,\ q)}\,d\varphi.$$

3. As above, we take $z = t^{-1}$ in (9) and replace q by q^2. Thus (9) becomes

$$\int_0^\infty \prod_1^\infty (1+q^{2n}xt)(1+q^n x^{-1}t^{-1})x^{-1}p(x)dx = \tag{13}$$

$$= \prod_{n=1}^{\infty}\frac{1-q^{2n}}{(1-q^{2n-1}t)(1-q^{2n-1}t^{-1})},$$

where again it is assumed that t satisfies (11). If we take $t = e^{2\pi iv}$, than right member of (13) becomes

$$\prod_{n=1}^{\infty}(1-q^{2n})^2/\theta_0(v,\ q),$$

where, in the usual notation,

$$\theta_0(v,\ q) = \theta_3\left(v+\frac{1}{2},\ q\right).$$

The integrand in the left member of (13) can be expressed in terms of $\theta_1(v)$, but it is perhaps simpler in its present form.

REFERENCES

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