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Finding the characteristic roots of symmetric matrices.

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Summary. - *For certain symmetric matrices similarity transformations have been tabulated to either diagonolize or transform the matrix into a form suitable for finding the characteristic roots.*

1. Introduction.

The problem of finding characteristic roots (c. n.) of matrices in general is laborious. However, if one takes advantage of symmetric properties of matrices, when such exists, the work is reduced considerably.

It is the purpose of this paper to tabulate similarity transformations $R = R^{-1}$, $(RR^{-1} = 1)$, such that a given matrix A when transformed into RAR^{-1} would be either a diagonal matrix, or one whose non-zero elements appear only above or only below the diagonal elements. It is well known that the diagonal elements of such transformed matrices are the c. n.

We shall make use of the following known results. For a more detailed discussion, the reader may consult any standard work on matrix theory.

DEFINITION - Matrices A and B are similar if and only if there exists a non-singular matrix P such that $PAP^{-1} = B$.

THEOREM 1.1 - Similar matrices have the same characteristic roots.

2. Symmetries.

We shall consider symmetries and « combinations » of symmetries of the following four types:

$$\begin{bmatrix} a & b \\ b & a \end{bmatrix}, \quad \begin{bmatrix} a & a \\ b & b \end{bmatrix}, \quad \begin{bmatrix} a & b & b \\ b & a & b \\ b & b & a \end{bmatrix}, \quad \begin{bmatrix} a & a & a \\ b & b & b \\ c & c & c \end{bmatrix}$$

CASE I

One finds that $R_2 = R_2^{-1} = \dfrac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ will diagonalize

$A_2 = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$ so that $R_2 A_2 R_2^{-1} = \begin{bmatrix} a+b & 0 \\ 0 & a-b \end{bmatrix}$. The c. n. of the transformed matrix are obviously $a + b$, $a - b$.

THEOREM 2.1 -

Let
$$A_1 = \left[\begin{array}{cc:cc} x & y & z & u \\ y & x & u & z \\ \hdashline z & u & x & y \\ u & z & y & x \end{array}\right]$$

(Note partitioned matrix A_4 has the same similarity structure as A_2) then $\exists$ and R_4 such that

$$R_4 A_4 R_4^{-1} \begin{bmatrix} x+y+z+u & 0 & 0 & 0 \\ 0 & x-y+z-u & 0 & 0 \\ 0 & 0 & x+y-z-u & 0 \\ 0 & 0 & 0 & x-y-z+u \end{bmatrix}.$$

PROOF: Consider the 4×4 matrix $\begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}$ where $P_1 = \begin{bmatrix} x+z & y+u \\ y+u & x+z \end{bmatrix}$ and $P_2 = \begin{vmatrix} x-z & y-u \\ y-u & x-z \end{vmatrix}$

Generalizing from the 2×2 case,

let $Q_4 = \dfrac{1}{\sqrt{2}} \left[\begin{array}{cc:cc} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ \hdashline 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{array}\right]$ and $T_4 = \dfrac{1}{\sqrt{2}} \left[\begin{array}{cc:cc} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ \hdashline 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{array}\right]$

and $R_4 = Q_4 T_4 = \dfrac{1}{\sqrt{4}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$. The reader will verify that

$R_4 = R_4^{-1}$ and that

$$R_4 A_4 R_4^{-1} = \begin{bmatrix} x+y+z+u & 0 & 0 & 0 \\ 0 & x-y+z-u & 0 & 0 \\ 0 & 0 & x+y-z-u & 0 \\ 0 & 0 & 0 & x-y-z+u \end{bmatrix}.$$

Corollary 2.1 Let $A_8 = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ where A_{ij} is 4×4 and $A_{11} = A_{22}$, $A_{21} = A_{12}$, and A_{ij} is of type A_4. (Note: As a partitioned matrix A_8 has same similarity structure as A_2) then $\exists$ an R_8 such that $R_8 A_8 R_8^{-1}$ will be a diagonal matrix.

Proof Again, generalizinig from the 4×4 case, let

$$Q_8 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

and T_8

$$\frac{1}{\sqrt{4}} \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & -1 & 0 & 1 \end{bmatrix}.$$

One computes

$$R_8 = Q_8 T_8 = \frac{1}{\sqrt{8}} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & +1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{bmatrix}.$$

It is easily verified that $R_8 = R_8^{-1}$ and that $R_8 A_8 R_8^{-1}$ is a diagonal matrix.

CASE II

If the matrix in question is of the type $B_2 = \begin{bmatrix} a & a \\ b & b \end{bmatrix}$, $R_2 = R_2^{-1}$ as defined in Case I, will transform B_2 into $R_2 B_2 R_2^{-1} = \begin{bmatrix} a+b & 0 \\ a-b & 0 \end{bmatrix}$, which is a desired form to find the c. n., $a+b$, 0.

If $B_4 = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$ where B_{ij} is a 2×2 matrix of type B_2 and

$B_{11} = B_{12}$; $B_{21} = B_{22}$ so that $B_4 = \left[\begin{array}{cc:cc} x & x & x & x \\ y & y & y & y \\ \hdashline z & z & z & z \\ u & u & u & u \end{array}\right]$, then $R_4 B_4 R_4^{-1} =$

$$\begin{bmatrix} x+y+z+u & 0 & 0 & 0 \\ x-y+z-u & 0 & 0 & 0 \\ x+y-z-u & 0 & 0 & 0 \\ x-y-z+u & 0 & 0 & 0 \end{bmatrix}.$$

CASE III

THEOREM 2.2

If C_3 is a matrix of the form $\begin{bmatrix} a & b & b \\ b & a & b \\ b & b & a \end{bmatrix}$, then $\exists$ an $S_3 = S_3^{-1}$ such that $S_3 C_3 S_3^{-1}$ is a matrix whose non-zero elements are all below the diagonal elements.

PROOF: Let $S_3 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & -3 & -2 \end{bmatrix}$.

One can verify that $S_3 = S_3^{-1}$ and that $S_3 C_3 S_3^{-1} =$

$$\begin{bmatrix} a+2b & 0 & 0 \\ 3b & a-b & 0 \\ -5b & 0 & a-b \end{bmatrix}.$$

COROLLARY 2.2

Let $C_9 = \begin{bmatrix} X & Y & Y \\ Y & X & Y \\ Y & Y & X \end{bmatrix}$ where $X = \begin{bmatrix} x & y & y \\ y & x & y \\ y & y & x \end{bmatrix}$ and $Y = \begin{bmatrix} z & u & u \\ u & z & u \\ u & u & z \end{bmatrix}$

then $\exists$ an $S_9 = S_9^{-1}$ such that $S_9 C_9 S_9^{-1}$ is a matrix whose non-zero elements are below the diagonal elements.

PROOF: Let

$$Q_9 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & -2 \end{bmatrix}$$

$$\text{and } T_9 = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & -3 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -3 & 0 & 0 & -2 \end{bmatrix}$$

$$\text{Now compute } S_9 = Q_9 T_9 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 1 & 0 & 2 & 1 & 0 & 2 & 1 \\ 0 & -3 & -2 & 0 & -3 & -2 & 0 & -3 & -2 \\ 0 & 0 & 0 & 2 & 2 & 2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 4 & 2 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & -6 & -4 & 0 & -3 & -2 \\ 0 & 0 & 0 & -3 & -3 & -3 & -2 & -2 & -2 \\ 0 & 0 & 0 & 0 & -6 & -3 & 0 & -4 & -2 \\ 0 & 0 & 0 & 0 & 9 & 6 & 0 & 6 & 4 \end{bmatrix}.$$

It is easily verified that $S_9 = S_9^{-1}$ and that $S_9 C_9 S_9^{-1}$ is a matrix in our desired form.

CASE IV

COROLLARY 2.3

If D_3 is a matrix of the form $D_3 = \begin{bmatrix} a & a & a \\ b & b & b \\ c & c & c \end{bmatrix}$ then $S_3 = S_3^{-1}$ will transform D_3 into

$$S_3 D_3 S_3^{-1} = \begin{bmatrix} a+b+c & 0 & 0 \\ 2b+c & 0 & 0 \\ -3b-2c & 0 & 0 \end{bmatrix}.$$

The c. n. here are obviously $a+b+c$, 0, 0.

3. Symmetry within a symmetry.

CASE I-II

THEOREM 3.1

If $A_4^* = \left[\begin{array}{cc:cc} x & x & z & z \\ y & y & u & u \\ \hdashline z & z & x & x \\ u & u & y & y \end{array}\right]$ (Note the partitioning indicates a symmetry of type II within a symmetry of type I), then $R_4 = R_4^{-1}$ will transform A_4^* into

$$R_4 A_4^* R_4^{-1} = \begin{bmatrix} x+y+z+u & 0 & 0 & 0 \\ x-y+z-u & 0 & 0 & 0 \\ 0 & 0 & x+y-z-u & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The reader will easily verify this.

CASE II-I

THEOREM 3.2

If $B_4^* = \left[\begin{array}{cc:cc} x & y & x & y \\ y & x & y & x \\ \hdashline z & u & z & u \\ u & z & u & z \end{array}\right]$. (Note a symmetry of type I within a symmetry of type II), then

$$R_4 B_4^* R_4^{-1} = \begin{bmatrix} x+y+z+u & 0 & 0 & 0 \\ 0 & x-y+z-u & 0 & 0 \\ x+y-z-u & 0 & 0 & 0 \\ 0 & x-y-z-u & 0 & 0 \end{bmatrix}.$$

CASE III-I

THEOREM 3.3

If $C_6^* = \begin{bmatrix} x & y & z & u & z & u \\ y & x & u & z & u & z \\ z & u & x & y & z & u \\ u & z & y & x & u & z \\ z & u & z & u & x & y \\ u & z & u & z & y & x \end{bmatrix}$ (Note the partitioning indicates a symmetry of type I within a symmetry of type III), then $S_6 C_6^* S_6^{-1}$ with

$$S_6 = S_6^{-1} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 \\ 0 & 0 & 2 & 2 & 1 & 1 \\ 0 & 0 & 2 & -2 & 1 & -1 \\ 0 & 0 & -3 & -3 & -2 & -2 \\ 0 & 0 & -3 & 3 & -2 & 2 \end{bmatrix},$$ will be a matrix

whose c. n. are the diagonal elements.

PROOF: Left to reader.

CASE III-II

THEOREM 3.4

If $C_6^{**} = \begin{bmatrix} x & x & z & z & z & z \\ y & y & u & u & u & u \\ z & z & x & x & z & z \\ u & u & y & y & u & u \\ z & z & z & z & x & x \\ u & u & u & u & y & y \end{bmatrix}$ (Note the partitioning indicates a symmetry of type II within a symmetry of type III) $S_6 C_6^{**} S_6^{-1}$ will be a matrix in our desired form.

PROOF: Left to reader.

CASE I-III

COROLLARY 24

If $A_6^{**}=\begin{bmatrix} x & y & y & z & u & u \\ y & x & y & u & z & u \\ y & y & x & u & u & z \\ z & u & u & x & y & y \\ u & z & u & y & x & x \\ u & u & z & y & y & x \end{bmatrix}$. E an $R_6^* = R_6^{*-1}$ such that $R_6^* A_6^{**} R_6^{*-1}$ is a matrix in our desired form

PROOF: Set $Q_6=\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & -3 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & -3 & -2 \end{bmatrix}$

and $T_6=\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{bmatrix}$

Compute $R_6^* = Q_6 T_6$.

The reader can easily verify that $R_6^* = R_6^{*-1}$ and that $R_6^* A_6^{**} R_6^{*-1}$ is a matrix whose non-zero elements are below the diagonal elements, and hence whose c. n. are the diagonal elements.

CASE IV-I

If $D_6^*=\begin{bmatrix} x & y & x & y & x & y \\ y & x & y & x & y & x \\ z & u & z & u & z & u \\ u & z & u & z & u & z \\ r & s & r & s & r & s \\ s & r & s & r & s & r \end{bmatrix}$, $S_6 D_6^* S_6^{-1}$ will also be a matrix in our desired form.

It seems quite likely that the results obtained in this paper this far can be extended to matrices of higher dimensions.