Bollettino
Unione Matematica Italiana

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Bollettino dell'Unione Matematica Italiana, Serie 3, Vol. **14** (1959), n.1, p. 78–81.

Zanichelli

<http://www.bdim.eu/item?id=BUMI_1959_3_14_1_78_0>

On the variation of the Fredholm resolvent.

Nota di RICHARD BELLMAN (a Santa Monica).

Sunto. - *Si ottiene un' equazione differenziale non lineare, che fornisce la variazione del nucleo di* FREDHOLM *con la lunghezza dell' intervallo.*

Summary. - *A nonlinear differential equation is obtained, showing the variation of the* FREDHOLM *kernel with the length of the interval.*

§ 1. - Introduction.

Consider the integral equation

$$u(x) = v(x) + \int_a^1 k(x,y)u(y)dy \tag{1}$$

whose solution, under suitable assumptions concerning the kernel $k(x,y)$, may be written in the form

$$u(x) = v(x) + \int_a^1 K(x,y,a)v(y)dy. \tag{2}$$

The function $K(x,y,a)$ is called the Fredholm resolvent. We have deliberately indicated the dependence upon a, since it is the variation of $K(x,y,a)$ as a function of the end-point a which we wish to discuss.

In a previous paper, [1], we considered the problem in the special case where $k(x,y)$ is symmetric, and such that the functional

$$J(u) = \int_a^1 u^2(x)dx - \int_a^1\int_a^1 k(x,y)u(x)u(y)dxdy \tag{3}$$

is positive definite for some range of a values such as $0 \leq a \leq 1$.

Using the functional equation technique of dynamic programming, we derived the result

$$(4) \qquad \frac{\partial K}{\partial a}(x,y,a) = - K(x,a,a)K(a,y,a).$$

In this paper, using a different technique, we shall prove

THEOREM 1. - *Let us assume that*

a) k(x,y) *is continuous in* x *and* y *for* $0 \leq x,\ y \leq 1$.

(5) *b) the solution of* (1) *may be written in the form given in* (2) *for* $0 \leq a \leq 1$.

Then (4) *holds for* $0 < a < 1$.

Using the same technique we shall give an alternative proof of a result of SCHIFFER [2] concerning the dependence of the GREEN's function of

$$(6) \qquad u'' + (p(x) + tq(x))u.$$

upon t.

§ 2. - Proof.

Without loss of generality, take $v(x)$ to be continuous over [0,1]. Then from (1,2) we obtain

$$(1) \qquad u_a(x) = K(x,a,a)v(a) + \int_a^1 \frac{\partial K}{\partial a}(x,y,a)v(y)dy,$$

while (1.1) yields

$$(2) \qquad u_a(x) = - k(x,a)u(a) + \int_a^1 k(x,y)u_a(y)dy.$$

Regarding this as an integral equation for the function $u_a(x)$, we see that $u_a(x)$ is given by

$$(3) \qquad u_a(x) = -k(x,a)u(a) + u(a)\int_a^1 K(x,y,a)k(y,a)dy .$$

Replacing $u(a)$ by the expression obtained from (1.2), we see that

$$(4) \qquad u_a(x) = -\left[v(a) + \int_a^1 K(a,y,a)v(y)dy \right] \cdot \left[k(x,a) + \int_a^1 K(x,y,a)k(y,a)dy \right].$$

Equating the expressions for $u_a(x)$ given in (1) and (4), we obtain the two results

$$(5) \qquad K(x,a,a) = k(x,a) + \int_a^1 K(x,y,a)k(y,a)dy ,$$

$$\frac{\partial K}{\partial a}(x,y,a) = -K(x,a,a)K(a,y,a) .$$

The first is readily derived from either FREDHOLM theory or the LIOUVILLE-NEUMANN solution, while the second is the result we wish to demonstrate.

§ 3. - A result of Schiffer.

Consider the equation

$$(1) \qquad u'' + (p(x) + tq(x))u = v(x) , \quad u(0) = u(1) = 0 ,$$

whose solution may be written

$$(2) \qquad u(x) = \int_0^1 K(x,y,t)v(y)dy .$$

We wish to study the variation of K with t.

We have

$$u_t(x) = \int_0^1 \frac{\partial K}{\partial t}(x,y,t)v(y)dy , \tag{3}$$

and, from (1),

$$\begin{aligned} u_t'' + (p(x) + tq(x))u_t &= -q(x)u , \\ u_t(0) = u_t(1) &= 0 . \end{aligned} \tag{4}$$

Thus, from (4),

$$\begin{aligned} u_t &= -\int_0^1 K(x,y,t)q(y)u(y)dy \\ &= -\int_0^1 K(x,y,t)q(y)\left[\int_0^1 K(y,z,t)v(z)dz\right]dy \\ &= -\int_0^1\left[\int_0^1 K(x,y,t)K(y,z,t)q(y)dy\right]v(z)dz . \end{aligned} \tag{5}$$

Equating (3) and (5), we see that

$$\frac{\partial K}{\partial t}(x,y,t) = -\int_0^1 K(x,z,t)K(z,y,t)q(z)dz . \tag{6}$$

This is the formula of SCHIFFER, [2]. It is clear that the same method can be used to study the GREEN's functions for a variety of boundary conditions.

REFERENCES

[1] R. BELLMAN, *Functional Equations in the Theory of Dynamic Programming — VII: An Integro-Differential Equation for the Fredholm Resolvent*, « Proc. Amer. Math. Soc. », Vol. 8 (1957), pp. 435-440.

[2] M. SCHIFFER, *Chap. 6, Modern Mathematics for Engineers*, E. F. Beckenbach, Editor, McGraw-Hill, 1956.