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On oscillators with large frequencies.

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Zanichelli

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On oscillators with large frequencies. [1]

Nota di PHILIP HARTMAN (a Baltimore)

Summary. - *A simple proof is given for the theorem of* ARMELLINI, SANSONE *and* TONELLI *that if* $q(t)$ *is continuous for* $t \geqq 0$, *is monotone,* $q(t) \to \infty$ *as* $t \to \infty$ *and log* $q(t)$ *is of "regular growth," then all solutions of* $d^2x/dt^2 + q(t)x = 0$ *tend to* 0 *as* $t \infty$.

Let $q(t)$ be a continuous, non-decreasing, unbounded function for $0 \leqq t < \infty$:

$$q > 0, \; dq \geqq 0, \; q \to \infty \text{ as } t \to \infty. \tag{1}$$

As is well known, every solution $x = x(t)$ of

$$x'' + q(t)x = 0 \tag{2}$$

is bounded; in fact the "conjugate energy" $E = E(t)$ belonging to $x(t)$,

$$E = x^2 + x'^2/q, \tag{3}$$

is non-increasing,

$$dE = - x'^2 dq/q^2 \leqq 0. \tag{4}$$

In answering a question raised by Biernacki, Milloux [2] has shown that (1) implies that (2) has a non-trivial ($\not\equiv 0$) solution satisfying

$$x \to 0, \text{ that is, } E \to 0, \text{ as } t \to \infty. \tag{5}$$

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A simple proof of Milloux's theorem and a generalization of it, involving a system of linear first order equations instead of (2), were given in [1].

It is not true that (1) implies (5) for *every* solution $x = x(t)$ of (2). In this direction, one has the following theorem:

(*) *Let* q(t) *be continuous for* $t \geqq 0$ *and satisfy* (1). *In addition, let* log q(t) *be of "regular growth". Then every solution* $x = x(t)$ *of* (2) *satisfies* (5).

This theorem was stated and proved in part by G. ARMELLINI; its proof was completed independently by G. SANSONE and L. TONELLI. For references, see [4], p. 61. For a generalization of (*), see [3].

According to ARMELLINI, a continuous, non-decreasing, unbounded function Q(t), where $t \geqq 0$, is called of *"irregular growth"* if, for every $\varepsilon > 0$, there exists an unbounded sequence of numbers $0 = t_0 < t_1 < \ldots$ such that if $C(n)$ and B are the open sets

$$(6) \qquad C(n) = \bigcup_{k=1}^{n} (t_{2k-1},\ t_{2k}),\ B = \bigcup_{k=0}^{\infty} (t_{2k},\ t_{2k+1}),$$

then

$$(7) \qquad \limsup_{n \to \infty} t_{2n}^{-1} \int_{C(n)} dt < \epsilon \text{ and} \int_{B} dQ < \infty .$$

If $Q(t)$ is not of irregular growth, then it is said to be of "regular growth".

The object of this note is to give a short, simple proof of (*), avoiding the awkward use for STURM's second comparison theorem in TONELLI's proof. (The argument below can be modified to give the result of OPIAL).

Proof of (*). For a non-trivial solution $x = x(t)$ of (2), let $\varphi(t)$ be the continuous function defined by

$$(8) \qquad \varphi(t) = \arctan\, q^{1/2}(t)x(t)/x'(t) \text{ and } 0 \leqq \varphi(0) < \pi .$$

Then, by (2),

$$(9) \qquad d\varphi = q^{1/2}dt + (\sin 2\varphi)\, dq/4q,$$

while (4) can be written as

$$-dE/E = (\cos^2 \varphi)\, d(\log q). \tag{10}$$

Suppose that (5) fails to hold for some solution $x=x(t)$; so that $E(\infty) > 0$ and $-\int^{\infty} dE/E < \infty$. Then

$$\int_0^{\infty} (\cos^2 \varphi)\, d(\log q) < \infty. \tag{11}$$

Let $0 < s_1 < s_2 < \dots$ be the sequence of positive zeros of $x = x(t)$ Since $d\varphi = q^{1/2}\, dt > 0$ at $t = s_n$, it follows that $\varphi(s_n) = n\pi$ for $n = 1, 2, \dots$ and that $n\pi < \varphi(t) < (n+t)\pi$ for $s_n < t < s_{n+1}$. If $0 < \mu < 1$ and $n > 1$, there exists a unique pair of t-values, $t = t_{2n}$ and $t = t_{2n+1}$ satisfying

$$s_{n-1} < t_{2n} < s_n < t_{2n+1} < s_{n+1},\ |\cos \varphi(t_{2n})| = |\cos \varphi(t_{2n+1})| = \mu$$

and

$$|\cos \varphi(t)| > \mu \text{ for } t_{2n} < t < t_{2n+1}. \tag{12}$$

(The numbers t_0, t_1 can be defined arbitrarily and will not be considered below). The positive numbers

$$\gamma_0 = \varphi(t_{2n}) - \varphi(t_{2n-1}),\ \gamma_1 = \varphi(t_{2n+1}) - \varphi(t_{2n}) \tag{13}$$

are independent of n. If $\varepsilon > 0$ is given and $\mu = \mu_\varepsilon$ is sufficiently small, then

$$0 < \gamma_0/\gamma_1 < \varepsilon\ (< 1). \tag{14}$$

Let $\Delta = \Delta(n)$ be the t-interval (t_{2n-1}, t_{2n}). By (9).

$$\int_\Delta dt \leq \int_\Delta q^{-1/2}\, d\varphi + \int_\Delta q^{-3/2} q'\, dt.$$

In the first integral on the right, replace $\varphi(t)$ by $\varphi(t)-\varphi(t_{2n})$ and integrate by parts. It follows, from $|\varphi(t)-\varphi(t_{2n})|\leqq\pi<4$ on Δ, that

$$\int_{\Delta} dt \leqq \gamma_0\, q^{-1/2}(t_{2n-1}) + 3\int_{\Delta} q^{-3/2}\, q' dt\,.$$

Similarly. if $\delta=\delta(n)$ is the t-intervall $(t_{2n-2},\ t_{2n-1})$,

$$\int_{\delta} dt \geqq \gamma_1\ q^{-1/2}(t_{2n-1}) - 3\int_{\delta} q^{-3/2} q' dt\,.$$

Hence,

$$\int_{\Delta} dt \leqq (\gamma_0/\gamma_1)\int_{\delta} dt + 3\int_{\Delta} q^{-3/2} q' dt + (3\,\gamma_0/\gamma_1)\int_{\delta} q^{-3/2}\, q' dt$$

and so, by (14),

$$\int_{\Delta} dt \leqq \varepsilon \int_{\Delta\cup\delta} dt + 3\int_{\Delta\cup\delta} q^{-3/2}\, q' dt\,.$$

Since $\int^{\infty} q^{-3/2} q' dt < \infty$, the set $C(n)$ in (6) satisfies the first part of (7).

It follows from (11) and (12) that the second part of (7) holds if $Q=\log\, q(t)$. Consequently, $\log\, q(t)$ is of irregular growth. Since this contradicts the hypothesis of (*), the assumption that (5) fails for some solution $x=x(t)$ is untenable. This proves (*).

REFERENCES

[1] P. HARTMAN, *On a theorem of Milloux*, « American Journal of Mathematics », vol. 70 (1948), pp. 395-399.

[2] H. MILLOUX, *Sur l'équation différentielle $x''+A(t)x=0$.* « Prace Matematyczno - Fizyczne », vol. 41 (1934), pp. 39-53.

[3] Z OPIAL, *Sur l'équation différentielle $u''+a(t)u=0$*, « Annales Polonici Matematici », vol. 5 (1958), pp. 77-93.

[3] G. SANSONE, *Equazioni differenziali nel campo reale*, vol. 2, Bologna (1949).