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Some biorthogonal q-polynomials in two variables.

Nota di LEONARD CARLITZ (ad Amherst, U. S. A)

Sunto. - *Si prova che i* q-*polinomi* $g_{m,n}(x, y)$, *definiti in* (1), *soddisfano ad una condizione di biortogonalità*

Summary. - *It is shown that the* q-*polynomials* $g_{m,n}(x, y)$ *defined in* (1) *below satisfy a biorthogonality relation.*

The writer has proved [2] that if $Q(x, y) = ax^2 + 2bxy + cy^2$ is a positive definite quadratic form, $|q| < 1$,

$$\theta(\Phi, \Psi) = \sum_{m, n=-\infty}^{\infty} q^{\frac{1}{2}Q(m, n)} e^{(m\Phi + n\Psi)i},$$

and

$$f_{m,n}(x, y) = \sum_{r=0}^{m} \sum_{s=0}^{n} (-1)^{r+s} q^{-Q(r, s)} \begin{bmatrix} m \\ r \end{bmatrix} \begin{bmatrix} n \\ s \end{bmatrix} q^{\frac{1}{2}r(r-1) + \frac{1}{2}s(s-1)} x^r y^s,$$

$$f'_{m,n}(x, y) = \sum_{r=0}^{m} \sum_{s=0}^{n} (-1)^{r+s} q^{-Q(r', s')} \begin{bmatrix} m \\ r \end{bmatrix} \begin{bmatrix} n \\ s \end{bmatrix} q^{\frac{1}{2}r(r-1) + \frac{1}{2}s(s-1)} x^{r'} y^{s'},$$

where r', s' satisfy

$$ar' + bs' = r, \qquad br' + cs' = s,$$

then

$$(2\pi)^{-2} \int_0^{2\pi} \int_0^{2\pi} f_{m,n}(e^{i\Phi}, e^{i\Psi}) f'_{m',n'}(e^{-i\Phi}, e^{-i\Psi}) \theta(\Phi, \Psi) d\Phi d\Psi =$$

$$= q^{-m-n} (q)_m (q)_n \delta_{mm'} \delta_{nn'}, \ .$$

where

$$(q)_m = (1-q) \cdots (1-q^m), \quad \begin{bmatrix} m \\ r \end{bmatrix} = \frac{(q)_m}{(q)_r (q)_{m-r}}.$$

In another paper [3] the writer has defined the polynomials

$$(1)\qquad g_{m,n}(x, y) = q^{\frac{1}{2}m(m-1)-\frac{1}{2}n(n-1)} \sum_{r=0}^{m} \sum_{s=0}^{n} (-1)^{m+n-r-s} \begin{bmatrix} m \\ r \end{bmatrix} \begin{bmatrix} n \\ s \end{bmatrix} \cdot q^{\frac{1}{2}r(r-1)+\frac{1}{2}s(s-1)-rs} x^r y^s,$$

$$(2)\qquad h_{m,n}(x, y) = \sum_{r=0}^{m} \sum_{s=0}^{n} (-1)^{r+s} \begin{bmatrix} m \\ r \end{bmatrix} \begin{bmatrix} n \\ s \end{bmatrix} q^{\frac{1}{2}r(r-1)+\frac{1}{2}s(s-1)+(m-r)(n-s)} x^{m-r} y^{n-s}.$$

The polynomial $g_{m,n}(x, y)$ evidently corresponds to the quadratic form $Q(x, y) = 2xy$, which is indefinite. Thus the above orthogonality relation does not apply. However, it is possible to obtain a result of this nature by employing a different weight function.

We define

$$\Theta(\Phi, \Psi) = \sum_{h,k=0}^{\infty} q^{(h+1)(k+1)} e^{(h\Phi+k\Psi)i},$$

Now consider

$$(2\pi)^{-2} \int_0^{2\pi}\int_0^{2\pi} g_{m,n}(q^{-m}e^{-i\Phi}, q^{-n}e^{-i\Psi}) g_{m',n'}(q^{-m'}e^{-i\Psi}, q^{-n'}e^{-i\Phi}) \Theta(\Phi, \Psi) d\Phi d\Psi =$$

$$= (-1)^{m+n+m'+n'} q^{-\frac{1}{2}m(m-1)-\frac{1}{2}n(n-1)-\frac{1}{2}m'(n'-1)-\frac{1}{2}n'(n'-1)} \cdot$$

$$\cdot \sum_{r,s} \sum_{r',s'} (-1)^{r+s+r'+s'} \begin{bmatrix} m \\ r \end{bmatrix} \begin{bmatrix} n \\ s \end{bmatrix} \begin{bmatrix} m' \\ r' \end{bmatrix} \begin{bmatrix} n' \\ s' \end{bmatrix} \cdot$$

$$\cdot q^{\frac{1}{2}r(r-1)+\frac{1}{2}s(s-1)+\frac{1}{2}r'(r'-1)+\frac{1}{2}s'(s'-1)-rs-r's'-rm-sn-r'm'-s'n'} \cdot$$

$$\cdot \sum_{h,k=0}^{\infty} q^{(h+1)(k+1)} (2\pi)^{-2} \int_0^{2\pi}\int_0^{2\pi} e^{(h-r-s')\Phi i+(k-s-r')\Psi i} d\Phi d\Psi.$$

$$= (-1)^{m+n+m'+n'} q^{-\frac{1}{2}m(n-1)-\frac{1}{2}n(n-1)-\frac{1}{2}m'(n'-1)-\frac{1}{2}n'(n'-1)} \cdot$$

$$\cdot \sum_{r,s} \sum_{r',s'} (-1)^{r+s+r'+s'} \begin{bmatrix} m \\ r \end{bmatrix} \begin{bmatrix} n \\ s \end{bmatrix} \begin{bmatrix} m' \\ r' \end{bmatrix} \begin{bmatrix} n' \\ s' \end{bmatrix} \cdot$$

$$\cdot\, q^{\frac{1}{2} r(r+1) + \frac{1}{2} s(s+1) + \frac{1}{2} r'(r'+1) + \frac{1}{2} s'(s'+1) - rm - sn - r'm' - s'n' + rr' + ss' + 1} =$$

$$= (-1)^{m+n+m'+n'} q^{-\frac{1}{2} m(m-1) - \frac{1}{2} n(n-1) - \frac{1}{2} m'(m'-1) - \frac{1}{2} n'(n'-1) + 1} \cdot$$

$$\cdot \sum_{r,r'} (-1)^{r+r'} \begin{bmatrix} m \\ r \end{bmatrix} \begin{bmatrix} m' \\ r' \end{bmatrix} q^{\frac{1}{2} r(r+1) + \frac{1}{2} r'(r'+1) - rm - r'm' + rr'} \cdot$$

$$\cdot \sum_{s,s'} (-1)^{s+s'} \begin{bmatrix} n \\ s \end{bmatrix} \begin{bmatrix} n' \\ s' \end{bmatrix} q^{\frac{1}{2} s(s+1) + \frac{1}{2} s'(s'+1) - sn - s'n' + ss'} .$$

But [1, formula (2. 7)]

$$\sum_{r=0}^{m} \sum_{r'=0}^{m'} (-1)^{r+r'} \begin{bmatrix} m \\ r \end{bmatrix} \begin{bmatrix} m' \\ r' \end{bmatrix} q^{\frac{1}{2} r(r+1) + \frac{1}{2} r'(r'+1) - rm - sn + rs} =$$

$$= (-1)^m q^{-\frac{1}{2} m(m-1)} (q)_m \delta_{mm'} .$$

Therefore we have

$$(2\pi)^{-2} \int_0^{2\pi} \int_0^{2\pi} g_{m,n}(q^{-m} e^{-i\Phi}, q^{-n} e^{-i\Psi}) g_{n',n'}(q^{-m'} e^{-i\Psi}, q^{-n'} e^{-i\Phi}) \Theta(\Phi, \Psi) d\Phi d\Psi =$$

$$= (-1)^{m+n} q^{1 - \frac{3}{2} m(m-1) - \frac{3}{2} n(n-1)} (q)_m (q)_n \delta_{mm'} \delta_{nn'},$$

so that we have obtained à biorthogonality relation.

The writer has been unable to find a similar relation for the polynomial $h_{m,n}(x, y)$ defined by (2).

REFERENCES

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[3] L. CARLITZ, *Some q-polynomials in two variables*, « Mathematische Nachrichten », to appear.