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Nota di RICHARD BELLMAN (a Santa Monica, U. S. A.)

Sunto. - *Si dimostra che se* $\log a(x, y)$ *è armonico si può cambiare la variabile indipendente in modo che l'equazione si trasformi in un'altra a coefficienti costanti.*

Summary. - *It is shown that under the assumption that log $a(x, y)$ is harmonic we can find a change of independent variable which reduces the equation to one with constant coefficients.*

1. Introduction.

In the study of the boundedness, stability and asymptotic behavior of the solutions of the second-order linear differential equation

$$u'' \pm a^2(t)u = 0, \tag{1}$$

an essential tool is the LIOUVILLE transformation

$$s = \int_0^t a(t_1)dt_1. \tag{2}$$

This is a $1 - 1$ transformation for large t if $a(t) > 0$ for all $t \geq t_0$. It transforms (1) into the equation

$$\frac{d^2u}{ds^2} + \frac{a'(t)}{a^2(t)} \frac{du}{ds} \pm u = 0. \tag{3}$$

If $a'(t)/a^2(t)$ is small in some sense, either as $t \to \infty$ because of the rate of increase of $a(t)$, or because $a(t)$ is slowly varying, we have an equation with almost-constant coefficients. A further change of variable

$$u = v/a(t)^{1/2} \tag{4}$$

reduces (3) to an equation of the form

$$\frac{d^2v}{ds^2} + (\pm 1 + b(s))v = 0. \tag{5}$$

From this equation, the *WKB* approximation follows immediately. For the details of these transformations and many further results, see CHAPTER 6 of our book, [1].

In studying the asymptotic behavior of the solutions of partial differential equations of the form

$$u_{xx} + u_{yy} \pm a^2(x, y)u = 0 \tag{6}$$

as $x, y \to \infty$, it is tempting to search for a transformation similar to that given in (2). In this paper, we will show that the desired transformation exists, and indeed does more than what might be expected, under certain favorable circumstances.

2. **Preliminary Calculations.**

Replacing x and y by two as yet unspecified independent variables s and t, we obtain the relations

$$\begin{aligned} u_{xx} &= u_{ss}s_x^2 + 2u_{st}s_xt_x + u_{tt}t_x^2 + u_ss_{xx} + u_tt_{xx} \\ u_{yy} &= u_{ss}s_y^2 + 2u_{st}s_yt_y + u_{tt}t_y^2 + u_ss_{yy} + u_tt_{yy}. \end{aligned} \tag{1}$$

The equation of (1.6) in the new variables has the form

$$\begin{aligned} u_{ss}(s_x^2 + s_y^2) &+ u_{tt}(t_x^2 + t_y^2) + 2u_{st}(s_xt_x + s_yt_y) \\ &+ u_s(s_{xx} + s_{yy}) + u_t(t_{xx} + t_{yy}) \\ &\pm a^2(x, y)u = 0. \end{aligned} \tag{2}$$

We wish to determine two functions of x and y. $s(x, y)$ and $t(x, y)$, such that the following relations hold:

$$\begin{aligned} s_x^2 + s_y^2 &= a^2(x, y) \\ t_x^2 + t_y^2 &= a^2(x, y) \\ s_xt_x + s_yt_y &= 0 . \end{aligned} \tag{3}$$

From the first two of these relations, we see that

$$\begin{aligned} s_x = a(x, y) \cos \varphi, \ t_x = a(x, y) \cos \psi, \\ s_y = a(x, y) \sin \varphi, \ t_y = a(x, y) \sin \psi, \end{aligned} \tag{4}$$

for two functions $\varphi(x, y)$ and $\psi(x, y)$.

The third relation, the orthogonality relation, requires that

(5) $$\varphi - \psi = \pm \pi/2.$$

Choosing $\psi = \varphi + \pi/2$, we have

(6) $$s_x = a(x, y) \cos \varphi, \; t_x = a(x, y) \sin \varphi,$$
$$s_y = a(x, y) \sin \varphi, \; t_y = - a(x, y) \cos \varphi.$$

The Jacobian of the transformation

(7) $$s = s(x, y)$$
$$t = t(x, y)$$

is thus $a^2(x, y)$. It follows that we wish to assume that $a^2(x, y) > 0$ for $x \geq x_0$, $y \geq y_0$.

It remains to determine under what conditions upon $a(x, y)$ there exists a function $\varphi(x, y)$ satisfying the desired relations.

3. Condition upon $a(x, y)$.

In order for (2.6) to hold, we must have

(1) $$(s_x)_y = (s_y)_x, \quad (t_x)_y = (t_y)_x,$$

or

(2) $$(a \cos \varphi)_y = (a \sin \varphi)_x, \quad (a \sin \varphi)_y = (- a \cos \varphi)_x.$$

A simple calculation shows that (2) is equivalent to the relations

(3) $$\varphi_x = a_y/a = \frac{\partial}{\partial y} (\log a),$$
$$\varphi_y = - a_x/a = - \frac{\partial}{\partial x} (\log a).$$

It follows that φ exists if, and only if,

(4) $$\frac{\partial^2}{\partial x^2} (\log a) + \frac{\partial^2}{\partial y^2} (\log a) = 0.$$

In other words, we must suppose that $\log a(x, y)$ *is a harmonic function.*

4. The Transformed Equation.

Having made this assumption, we reap the bonus that

$$(1)\qquad \begin{aligned} s_{xx} &= a_x \cos\varphi - a\varphi_x \sin\varphi = a_x \cos\varphi - a_y \sin\varphi, \\ s_{yy} &= a_y \sin\varphi + a\varphi_y \cos\varphi = a_y \sin\varphi - a_x \cos\varphi, \end{aligned}$$

whence

$$(2)\qquad s_{xx} + s_{yy} = 0,$$

and similarly

$$(3)\qquad t_{xx} + t_{yy} = 0.$$

Thus the eqnation in terms of the new variables is

$$(4)\qquad u_{ss} + u_{tt} \pm u = 0.$$

5. Discussion.

In connection with the study of the asymptotic behavior of the solutions of (1.6), we need only demand that the principal term of log $a(x, y)$ be harmonic. Using this principal term and carrying through the foregoing transformation. we will obtain an equation of the form

$$(1)\qquad u_{ss} + u_{tt} + (\pm 1 + b(s, t))u = 0.$$

We shall discuss these matters elsewhere.

Observe that for $a(x, y) = e^{kxy}$, the foregoing transformation yields various reasonably explicit solutions of the equation

$$(2)\qquad u_{xx} + u_{yy} \pm e^{2kxy}u = 0.$$

REFERENCE

[1] Bellman, R., *Stability Theory of Differential Equations*, McGraw-Hill Book Company, Inc., New York, 1954.