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On sequences of complex terms defined by iteration.

Nota di A. G. AZPEITIA (ad Amherst, U. S. A.)

Sunto. - *Si dimostra la convergenza di un certo tipo di successioni di numeri complessi, definite mediante una data successione di funzioni complesse di più variabili complesse.*

Summary. - *The convergence of a certain type of sequences of complex numbers defined in terms of a given sequence of complex functions of several variables, is established.*

The purpose of this note is to establish a generalization of a previously proved theorem (See Ref. 1 at the end of this paper). The statement that we will prove is as follows:

THEOREM. – *Let* $\{ f_n(z_1, z_2, \dots, z_p) \}$, $(n = 1, 2, \dots)$ *be a sequence of complex functions of the* p *complex variables* $z_1, z_2, \dots, z_p$ *defined for* $z_i \in D$, $(i = 1, 2, \dots, p)$, *where* D *is a convex region of the complex plane* (z), *and let us assume that*

1) *For* $z_i \in D$ *the sequence* $\{ f_n(z_1, z_2, \dots, z_p)$ *converges uniformly to a continuous function* $f_0(z_1, z_2, \dots, z_p)$.

2) *If* $z_i \in D$ *then, for any* $n \geqq 0$, *the point* $w_n = f_n(z_1, z_2, \dots, z_p)$ *belongs to the closed convex hull* R_n *of the set of points* z_i *and furthermore,* w_n *is different of the extreme points of* R_n, *unless all the* z_i *are equal in which case* $w_n = z_i$.

Under these conditions, every complex sequence $\{ a_n \}$, $(n = 1, 2, \dots)$ *defined by*

$$a_n = f_n(a_{n-p}, a_{n-p+1}, \dots, a_{n-1})$$

with the initial terms $a_1, a_2, \dots, a_p$ *arbitrarily chosen in* D *is convergent.*

Of course, if the f_n, $(n = 1, 2, \dots)$ are continuous functions, the continuity of the limit function f_0 is guaranteed by the uniform convergence and can be removed from among the initial assumptions. Also the property $f_n(z, z, \dots, z) = z$, $(n = 0, 1, 2, \dots)$ would be a consequence of the continuity of the f_n and the first part of assumption 2).

The particular case of this theorem, when all the functions f_n are identical, was established in the previously mentioned paper (Ref. 1), applying the following property of convex polygonal sets that we will use as well for the present generalization.

(I) *If the bounded closed sets* $P_n \subset (z)$, $(n = 1, 2, \dots)$ *are convex polygons with* p *or less vertices and such that* $P_n \supset P_{n+1}$, *then* *the set* $\mathrm{P} = \bigcap_{n=1}^{\infty} \mathrm{P_n}$ *is a bounded closed convex polygon with* p *or less vertices (or as a particular case, a single point), and every vertex of* P *is limit point of sequences of points formed by vertices of the polygons* [1].

To demonstrate the THEOREM, we define a sequence of polygons $\{P_n\}$ which are the closed convex hulls of the sets of points E_n with elements $(a_{n-p}, a_{n-p+1}, \dots, a_{n-1})$. Obviously, these P_n fulfill the conditions requested in (I) and the Theorem will be established if we prove that the intersection $P = \bigcap_{n=1}^{\infty} P_n$ consists of a single point.

We know, because of (I), that P is a polygon with $p' \leq p$ vertices h_j, $(j = 1, 2, \dots, p')$. Now, from the sequence $\{a_n\}$ we can select a partial sequence $\{a_{n'}\} \subset \{a_n\}$ such that $\lim_{n'=\infty} a_{n'} = h_1$, (where n' is therefore a sequence of strictly increasing positive and possibly non consecutive integers). The corresponding sequence $\{P_{n'}\}$ of convex hulls of the sets $(a_{n'-p}, a_{n'-p+1}, \dots, a_{n'-1})$ has intersection P, and by virtue of (I) it is possible to select a partial sequence $\{P_{n''} \subset \{P_{n'}\}$ such that the sequence $\{a_{n''-1}\}$ has a unique limit α_1.

Again from the new sequence of polygons $P_{n''}$, convex hulls of $(a_{n''-p}, a_{n''-p+1}, \dots, a_{n''-1})$ we select $\{P_{n'''}\} \subset \{P_{n''}\}$ such that $\lim_{n'''=\infty} a_{n'''-2} = \alpha_2$ exists, and so on. After repeating this process p times, we obtain a sequence of indices $n^{(p+1)}$ such that

$$\lim_{n^{(p+1)}=\infty} a_{n^{(p+1)}} = h_1 \qquad \text{and} \qquad \lim_{n^{(p+1)}=\infty} a_{n^{(p+1)}-i} = \alpha_i, \qquad (i = 1, 2, \dots, p)$$

and that the polygons $P_{n^{(p+1)}}$, convex hulls of the sets $(a_{n^{(p+1)}-p}, a_{n^{(p+1)}-p+1}, \dots, a_{n^{(p+1)}-1})$, verify the assumptions of (I) and have intersection P. Therefore, the set S, with elements $\alpha_1, \alpha_2, \dots, \alpha_p$, contains all the points h_j and also, since every α_i belongs to all the P_n, it is $S \subset P$.

At the same time, from the definition of the sequence $\{a_n\}$, we obtain

$$(1) \qquad h_1 = \lim_{n^{(p+1)}=\infty} a_{n^{(p+1)}} = \\ = \lim_{n^{(p+1)}=\infty} f_{n^{(p+1)}}(a_{n^{(p+4)}-p}, a_{n^{(p+1)}-p+1}, \dots, a_{n^{(p+1)}-1})$$

[1] The proof is given in Ref. 1, Lemma and Theorem II.

and from this it follows, as we will prove later, that

$$h_1 = f_0(\alpha_1, \alpha_2, \dots, \alpha_p) \tag{2}$$

and since h_1 coincides with at least one α_i, the assumption (2) applied to the function f_0, implies

$$h_1 = \alpha_1 = \alpha_2 = \dots = \alpha_p$$

and consequently the polygon P consists of one single point, and the proof will be completed as soon as we justify the legitimacy of taking limits in (1) to obtain (2).

This is, of course, immediate consequence of the theorem of the continuous convergence generalized to a complex space of p dimensions (see Ref. 2, t. I. p. 179). However, it is easy to give a direct proof as follows :

For the sake of brevity, let us denote by $\{m\}$ the sequence $\{n^{(p+1)}\}$.

The numbers m are therefore strictly increasing positive integers.

The continuity of f_0 implies that for every $\varepsilon > 0$ there exists a number N_1 such that if $m > N_1$, we have

$$| f_0(a_{m-p}, a_{m-p+1}, \dots, a_{m-1}) - f_0(\alpha_1, \alpha_2, \dots, \alpha_p) | < \varepsilon/2. \tag{3}$$

On the other hand, the uniform convergence of $\{f_m\}$ means that it exists N_2 such that for $m > N_2$ it is

$$| f_m(z_1, z_2, \dots, z_p) - f_0(z_1, z_2, \dots, z_p) | < \varepsilon/2 \quad \text{for all} \quad z_i \in D$$

and in particular

$$| f_m(a_{m-p}, a_{m-p+1}, \dots, a_{m-1}) - f_0(a_{m-p}, a_{m-p+1}, \dots, a_{m-1}) | < \varepsilon/2 \tag{4}$$

and by comparison of (3) and (4), we obtain

$$| f_m(a_{m-p}, a_{m-p+1}, \dots, a_{m-1}) - f_0(\alpha_1, \alpha_2, \dots, \alpha_p) | < \varepsilon$$

for every $m > \max(N_1, N_2)$ and, consequently, the theorem is proved.

REFERENCES

[1] A. G. Azpeitia, *Convergence of sequences of complex terms defined by iteration* Proc. Am. Math. Soc., v. 9, no. 3, June 1958, pp. 428-432.

[2] C. Caratheodory, *Theory of functions of a complex variable*, Chelsea Publishing Co., New York, 1954.