Suresh Chandra Arya

Abelian theorem for a generalised Stieltjes transform.

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Abelian theorem for a generalised Stieltjes transform.

Nota di SURESH CHANDRA ARYA (a Nainital, India)

Sunto. - *In questa comunicazione è mostrato che se la trasformazione*

$$f(s) = \frac{\Gamma(2m+1)}{\Gamma(m-k+3/2)} \frac{1}{s} \int_0^\infty F\left[\begin{matrix} 2m+1,\ 1 \\ m-k+3/2 \end{matrix}; -\frac{t}{s}\right] d\alpha(t)$$

che è una generalizzazione della trasformazione di STIELTJES, *converge, allora in certe condizioni per qualsiasi costante* A

$$\overline{\lim_{\substack{s \to 0+ \\ s \to \infty}}} \left| \frac{\Gamma(m-k+3/2)}{\Gamma(2m+1)} s^{\Upsilon-m-k+1/2} f(s) - A \right| \leq \overline{\lim_{\substack{t \to 0+ \\ t \to \infty}}} \left| \frac{\alpha(t)}{ct^{m+k+1/2-\Upsilon}} - A \right|,$$

dove c *è una certa costante.*

Summary. - *In this paper it is shown that if the transform*

$$f(s) = \frac{\Gamma(2m+1)}{\Gamma(m-k+3/2)} \frac{1}{s} \int_0^\infty F\left[\begin{matrix} 2m+1,\ 1 \\ m-k+3/2 \end{matrix}; -\frac{t}{s}\right] d\alpha(t)$$

which is a generalization of STIELTJES *transform, converges, then under certain conditions for any constant* A

$$\overline{\lim_{\substack{s \to 0+ \\ s \to \infty}}} \left| \frac{\Gamma(m-k+3/2)}{\Gamma(2m+1)} s^{\Upsilon-m-k+1/2} f(s) - A \right| \leq \overline{\lim_{\substack{t \to 0+ \\ t \to \infty}}} \left| \frac{\alpha(t)}{ct^{m+k+1/2-\Upsilon}} - A \right|,$$

where c *is a certain constant.*

1. Introduction.

If we iterate the LAPLACE Transform, i. e. if we take

$$f_0(s) = \int_0^\infty e^{-st}\psi(t)dt,$$

where

$$\psi(s) = \int_0^\infty e^{-st}\varphi(t)dt,$$

then

$$f_0(s) = \int_0^\infty \frac{\varphi(t)}{s+t} dt \tag{1.1}$$

and (1.1) is referred to as the STIELTJES Transform. On replacing $\varphi(t)dt$ by $d\alpha(t)$ we get the more general case of (1.1) in the form

$$f_0(s) = \int_0^\infty \frac{d\alpha(t}{s+t}. \tag{1.2}$$

VARMA [3] has given a generalisation of LAPLACE integral in the form

$$f(s) = \int_0^\infty (st)^{m-1/2} e^{-(st)/2} W_{k,\,m}(st)\psi)t)dt. \tag{1.3}$$

He has also shown that if $f(s)$ be the transform of $\psi(t)$ in this generalised sense and $\psi(s)$ to be the ordinary LAPLACE transform of $\varphi(t)$, we get

$$f(s) = \frac{\Gamma(2m+1)}{\Gamma(m-k+3/2)} \frac{1}{s} \int_0^\infty F\left[\begin{matrix}2m+1,\ 1\\ m-k+3/2\end{matrix};\ -\frac{t}{s}\right] \varphi(t)dt \tag{1.4}$$

where $Re\,(2m+1) > 0$.

If $k + m = \frac{1}{2}$, the hypergeometric function degenerates into a binomial expression and we get the ordinary STIELTJES Transform. If $k - m = \frac{1}{2}$, and $2m + 1 = \rho$ then also the hypergeometric function degenerates into another binomial expression and we get

$$\theta(s) = \frac{1}{\Gamma(\rho)} s^{-\rho+1} f(s) = \int_0^\infty \frac{\varphi(t)dt}{(s+t)^\rho} \tag{1.5}$$

which is a generalised form of STIELTJES Transform. On replacing $\varphi(t)dt$ by $d\alpha(t)$ in (1.4) and (1.5) we get the more general cases in the forms

$$f(s) = \frac{\Gamma(2m+1)}{\Gamma(m-k+3/2)} \frac{1}{s} \int_0^\infty F\left[\begin{matrix}2m+1,\ 1\\ m-k+3/2\end{matrix};\ -\frac{t}{s}\right] d\alpha(t) \tag{1.6}$$

and

$$\theta(s) = \int_0^\infty \frac{d\alpha(t)}{(s+t)^\rho}. \tag{1.7}$$

In this paper we will give an Abelian Theorem for the transform (1.6). In what follows we will assume that $\alpha(x)$ is a normalised function of bounded variation, and s is a real variable.

2. Abelian Theorem.

THEOREM A. – If (1.6) converges for $s > 0$, and if

(i) $Re\,(2m+1) > 0$,

(ii) $m - k + \frac{3}{2} \neq 0, -1, -2, \ldots$

and

(iii) $Re\left(m - k + \frac{1}{2} + \gamma\right) > 0$, $Re\left(\frac{1}{2} - m - k + \gamma\right) > 0$,

$Re\left(m + k + \frac{3}{2} - \gamma\right) > 0$, then for any constant A

$$(2.1)\quad \overline{\lim_{\substack{s \to 0+ \\ s \to \infty}}} \left| \frac{\Gamma(m-k+3/2)}{\Gamma(2m+1)} s^{\gamma-m-k+1/2} f(s) \quad A \right| \leq \overline{\lim_{\substack{t \to 0+ \\ t \to \infty}}} \left| \frac{\alpha(t)}{c t^{m+k+1/2-\gamma}} - A \right|,$$

where

$$(2.2)\quad c = \frac{\Gamma(2m+1)\Gamma(-2k+1+\gamma)}{\Gamma(m-k+3/2)\Gamma(m-k+1/2+\gamma)\Gamma(1/2-m-k+\gamma)\Gamma(m+k+3/2-\gamma)}$$

PROOF. – Integrating by parts, we have

$$f(s) = \frac{\Gamma(2m+1)}{\Gamma(m-k+3/2)} \frac{1}{s} \left[\alpha(t) F \left[\begin{matrix} 2m+1,\ 1 \\ m-k+3/2 \end{matrix} ;\ -\frac{t}{s} \right] \right]_0^\infty +$$

$$+ \frac{\Gamma(2m+2)}{\Gamma(m-k+5/2)} \frac{1}{s^2} \int_0^\infty F \left[\begin{matrix} 2m+2,\ 2 \\ m-k+5/2 \end{matrix} ;\ -\frac{t}{s} \right] \alpha(t)\,dt =$$

$$= \frac{\Gamma(2m+2)}{\Gamma(m-k+5/2)} \frac{1}{s^2} \int_0^\infty F \left[\begin{matrix} 2m+2,\ 2 \\ m-k+5/2 \end{matrix} ;\ -\frac{t}{s} \right] \alpha(t)\,dt,$$

for $\alpha(0) = 0$ and

$$\alpha(t) F \left[\begin{matrix} 2m+1,\ 1 \\ m-k+3/2 \end{matrix} ;\ -\frac{t}{s} \right] = o(1) \qquad (t \to \infty)$$

since (1.6) converges [1].

Now

$$(2.3)\quad s^{\gamma-m-k-3/2} \frac{(2m+1)}{(m-k+3/2)} \int_0^\infty t^{m+k+1/2-\gamma} F \left[\begin{matrix} 2m+2,\ 2 \\ m-k+5/2 \end{matrix} ;\ -\frac{t}{s} \right] dt =$$

$$=\frac{\Gamma(m-k+3/2)\Gamma(m-k+1/2+\gamma)\Gamma(1/2-m-k+\gamma)\Gamma(m+k+3/2-\gamma)}{\Gamma(2m+1)\Gamma(-2k+1+\gamma)}$$

provided that $Re\,(m-k+1/2+\gamma)>0$, $Re\left(\frac{1}{2}-m-k+\gamma\right)>0$, $Re\,(m+k+3/2-\gamma)>0$, and $m-k+\frac{5}{2}\neq 0,\ -1,\ -2,\ldots$ since [2, page 79 (4)]

$$\int_0^\infty z^{-s-1}F[a,\ b\ ;\ c\ ;\ -z]dz=\frac{\Gamma(a+s)\Gamma(b+s)\Gamma(c)\Gamma(-s)}{\Gamma(a)\Gamma(b)\Gamma(c+s)}.$$

Let us define c by (2.2).

Therefore

$$J\equiv\frac{\Gamma(m-k+3/2)}{\Gamma(2m+1)}\,s^{\gamma-m-k+1/2}f(s)-A,$$

$$=\frac{(2m+1)}{(m-k+3/2)}\,s^{\gamma-m-k-3/2}\int_0^\infty F\left[\begin{matrix}2m+2,\ 2\\ m-k+5/2\end{matrix};\ -\frac{t}{s}\right]\left\{\alpha(t)-Act^{m+k+1/2-\gamma}\right\}dt,$$

or

$$|\,J\,|\leq \underset{0<t\leq T}{\text{l. u. b.}}\left|\frac{\alpha(t)}{ct^{m+k+1/2-\gamma}}-A\right|\cdot$$

$$\cdot\left|\frac{(2m+1)c}{(m-k+3/2)}\int_0^T\left(\frac{t}{s}\right)^{m+k+1/2-\gamma}F\left[\begin{matrix}2m+2,\ 2\\ m-k+5/2\end{matrix};\ -\frac{t}{s}\right]d(t/s)\right|+$$

$$+\left|\frac{(2m+1)}{m-k+3/2)}\,s^{\gamma-m-k-3/2}\int_T^\infty F\left[\begin{matrix}2m+2,\ 2\\ m-k+5/2\end{matrix};\ -\frac{t}{s}\right]\left\{\alpha(t)-Act^{m+k+1/2-\gamma}\right\}dt\right|\leq$$

$$\leq \underset{0<t\leq T}{\text{l. u. b.}}\left|\frac{\alpha(t)}{ct^{m+k+1/2-\gamma}}-A\right|+$$

$$+\left|\frac{(2m+1)}{(m-k+3/2)}\,s^{\gamma-m-k-3/2}\int_T^\infty F\left[\begin{matrix}2m+2,\ 2\\ m-k+5/2\end{matrix};\ -\frac{t}{s}\right]\alpha(t)dt\right|+$$

$$+\left|Ac\frac{(2m+1)}{(m-k+3/2)}\int_T^\infty\left(\frac{t}{s}\right)^{m+k+1/2-\gamma}F\left[\begin{matrix}2m+2,\ 2\\ m-k+5/2\end{matrix};\ -\frac{t}{s}\right]d(t/s)\right|.$$

But

$$\int_T^\infty \left(\frac{t}{s}\right)^{m+k+1/2-\gamma} F\left[\begin{matrix}2m+2,\ 2\\ m-k+5/2\end{matrix};\ -\frac{t}{s}\right] d(t/s) =$$

$$= \int_{T/s}^\infty v^{m+k+1/2-\gamma} F\left[\begin{matrix}2m+2,\ 2\\ m-k+5/2\end{matrix};\ -v\right] dv$$

which tends to zero as $s \to 0+$.

Let us set

$$U \equiv F\left[\begin{matrix}2m+2,\ 2\\ m-k+5/2\end{matrix};\ -z\right]$$

then [2, page 63]

$$U = 0(z^{-2}) \qquad (z \to \infty) \tag{2.4a}$$

if (*i*) $2m$ is a positive integer,

or (*ii*) $2m \neq 0$ or a positive integer, $k \pm m \neq 1/2$ and $Re\, m \geqq 0$,

or (*iii*) $k + m = 1/2$,

or (*iv*) $2m = 0$, $-k + 1/2 = 0$;

$$U = 0\,(z^{-2m-2}) \qquad (z \to \infty) \tag{2.4b}$$

if (*i*) $k - m = 1/2$,

or (*ii*) $2m \neq 0$ or a positive integer and $Re\, m < 0$;

and

$$U = 0(z^{-2} \log z) \qquad (z \to \infty) \tag{2.4c}$$

if $2m = 0$ and $-k + 1/2 \neq 0$.

Now take s so small that under conditions satisfying (2.4*a* we have

$$\left| F\left[\begin{matrix}2m+2,\ 2\\ m-k+5/2\end{matrix};\ -\frac{t}{s}\right]\right| \leqq M\left(\frac{t}{s}\right)^{-2}.$$

Therefore, for small s

$$\left| s^{\gamma-m-k-3/2} \int_T^\infty F\left[\begin{matrix}2m+2,\ 2\\ m-k+5/2\end{matrix};\ -\frac{t}{s}\right] \alpha(t)dt \right| \leqq \tag{2.5a}$$

$$\leqq \left| s^{\gamma-m-k+1/2} M \int_T^\infty t^{-2}\alpha(t)dt \right|.$$

The integral $\int_T^\infty t^{-2}\alpha(t)dt$ is convergent, since

$$\alpha(t) = o(t) \qquad (t \to \infty)$$

under the same conditions. Therefore the left hand side of (2.5*a*) tends to zero as s tends to zero if $Re(\gamma - m - k + 1/2) > 0$, which we have supposed, is already satisfied.

Similarly, also for small s

$$(2.5b) \qquad \left| s^{\gamma-m-k-3/2} \int_T^\infty F\left[\begin{matrix}2m+2,\ 2\\ m-k+5/2\end{matrix};\ -\frac{t}{s}\right]\alpha(t)dt \right| \leqq$$

$$\leqq \left| s^{\gamma+m-k+1/2} M_1 \int_T^\infty t^{-2m-2}\alpha(t)dt \right|$$

under the conditions (2.4*b*) and the integral $\int_T^\infty t^{-2m-2}\alpha(t)dt$ is convergent since $\alpha(t) = o(t^{2m+1})$ $(t \to \infty)$ under the same conditions. Therefore the left hand side of (2.5*b*) tends to zero if $Re(\gamma + m - - k + 1/2) > 0$, which, we have supposed, is already satisfied.

Again, similarly for small s

$$(2.5c) \qquad \left| s^{\gamma-m-k-3/2} \int_T^\infty F\left[\begin{matrix}2m+2,\ 2\\ m-k+5/2\end{matrix};\ -\frac{t}{s}\right]\alpha(t)dt \right| \leqq$$

$$\leqq \left| s^{\gamma-m-k+1/2} M_2 \int_T^\infty t^{-2} \log (t/s)\alpha(t)dt \right|$$

under the conditions satisfying (2.4*c*) and the integral $\int_T^\infty t^{-2} \log\left(\frac{t}{s}\right)\alpha(t)dt$ is convergent since $\alpha(t) = o(t/\log t)$ $(t \to \infty)$ under the same conditions. Therefore the left hand side of (2.5*c*) tends to zero if $Re(\gamma - m - k + 1/2) > 0$, which is already satisfied.

Hence

$$s^{\gamma-m-k-3/2} \int_T^\infty F\left[\begin{matrix}2m+2,\ 2\\ m-k+5/2\end{matrix};\ -\frac{t}{s}\right]\alpha(t)dt$$

tends to zero as s tends to zero under the conditions stated.

Therefore

$$\overline{\lim_{s\to 0+}}\left| s^{\gamma-m-k+1/2}f(s)\frac{\Gamma\, 2m+1)}{\Gamma(m-k+3/2)}-A\right| \leqq \underset{0<t\leqq T}{\text{l. u. b.}}\left|\frac{\alpha(t)}{ct^{m+k+1/2-\gamma}}-A\right|.$$

Since T is arbitrary, we have

$$\overline{\lim_{s\to 0+}}\left| s^{\gamma-m-k+1/2}f(s)\,\frac{\Gamma(m-k+3/2)}{\Gamma(2m+1)}-A\right| \leqq \overline{\lim_{t\to 0+}}\left|\frac{\alpha(t)}{ct^{m+k+1/2-\gamma}}-A\right|.$$

Also, as before

$$\left| s^{\gamma-m-k+1/2}f(s)\,\frac{\Gamma(m-k+3/2)}{\Gamma(2m+1)}-A\right| \leqq \underset{T\leqq t<\infty}{\text{l. u. b.}}\left|\frac{\alpha(t)}{ct^{m+k+1/2-\gamma}}-A\right|\cdot$$

$$\cdot\left|\frac{(2m+1)c}{m-k+3/2)}\int_T^\infty\left(\frac{t}{s}\right)^{m+k+1/2-\gamma}F\left[\begin{matrix}2m+2,\ 2\\ m-k+5/2\end{matrix};\ -\frac{t}{s}\right]d\left(\frac{t}{s}\right)\right|+$$

$$+\left|\frac{(2m+1)}{(m-k+3/2)}\,s^{\gamma-m-k-3/2}\int_0^T F\left[\begin{matrix}2m+2,\ 2\\ m-k+5/2\end{matrix};\ -\frac{t}{s}\right]\alpha(t)dt\right|+$$

$$+\left|A\,\frac{(2m+1)c}{(m-k+3/2)}\int^T\left(\frac{t}{s}\right)^{m+k+1/2-\gamma}F\left[\begin{matrix}2m+2,\ 2\\ m-k+5/2\end{matrix};\ -\frac{t}{s}\right]d\left(\frac{t}{s}\right)\right|.$$

The last integral equals

$$\int_0^{T/s} v^{m+k+1/2-\gamma}F\left[\begin{matrix}2m+2,\ 2\\ m-k+5/2\end{matrix};\ -v\right]dv,$$

which tends to zero as s tends to infinity.

Also, for given T, the integral

$$s^{\gamma-m-k-3/2}\int_0^T F\left[\begin{matrix}2m+2,\ 2\\ m-k+5/2\end{matrix};\ -\frac{t}{s}\right]\alpha(t)dt$$

tends to zero as s tends to infinity if $Re(\gamma-m-k-3/2)<0$, which we have supposed in the theorem.

Therefore

$$\overline{\lim_{s\to\infty}} \left| s^{\gamma-m-k+1/2} f(s) \frac{\Gamma(m-k+3/2)}{\Gamma(2m+1)} - A \right| \leqq \underset{T\leq t<\infty}{\text{l. u. b.}} \left| \frac{\alpha(t)}{c t^{m+k+1/2-\gamma}} - A \right|$$

or since T is arbitrary, we have

$$\overline{\lim_{s\to\infty}} \left| s^{\gamma-m-k+1/2} f(s) \frac{\Gamma(m-k+3/2)}{\Gamma(2m+1)} - A \right| \leqq \overline{\lim_{t\to\infty}} \left| \frac{\alpha(t)}{c t^{m+k+1/2-\gamma}} - A \right|$$

Corollary 1. - If $k - m = 1/2$ and $2m + 1 = \rho$, we have the following theorem due to Widder [4, page 183].

Theorem B. - If (1.7) converges for $s > 0$ and $0 < \gamma \leqq \rho$, then for any constant A

$$\overline{\lim_{\substack{s\to 0+\\ s\to\infty}}} \left| s^{\gamma}\theta(s) - A \right| \leqq \overline{\lim_{\substack{t\to 0+\\ t\to\infty}}} \left| \frac{\alpha(t)}{c_1 t^{\rho-\gamma}} - A \right| \tag{2.6}$$

where

$$c_1 = \frac{\Gamma(\rho)}{\Gamma(\gamma)\Gamma(\rho-\gamma+1)}.$$

Corollary 2. - If $k + m = 1/2$, we have the following theorem.

Theorem C. - If (1.2) converges for $s > 0$, and if $0 < \gamma \leqq 1$, then for any constant A

$$\overline{\lim_{\substack{s\to 0+\\ s\to\infty}}} \left| s^{\gamma} f_0(s) - A \right| \leqq \overline{\lim_{\substack{t\to 0+\\ t\to\infty}}} \left| \frac{\alpha(t)}{c_2 t^{1-\gamma}} - A \right| \tag{2.7}$$

where

$$c_2 = \frac{1}{\Gamma(\gamma)\Gamma(2-\gamma)}.$$

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