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On Bessel polynomials.

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On Bessel polynomials.

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Sunto. - *Si ottengono alcuni nuovi risultati sui polinomi di* BESSEL *usando l'equazione di* TRUESDELL.

Summary. - *Some new results are got for* BESSEL *polynomials by the use of* TRUESDELL'*s* F*-equation.*

It is of great interest to note that a particular form of the BESSEL polynomials follow TRUESDELL's F-equation [4]. Though there has been scores of papers, of late, on this polynomial, this point appears to have escaped notice. In the present note it is attempted to discuss this phase of the problem. Only a few results are presented here and many more may follow by direct application of TRUESDELL's general results.

W. A. AL-SALAM [2] has given the connection between the BESSEL polynomials y_n (z, a, b) and LAGUERRE polynomials $L_n{}^{\nu}$ (vide Ref. [2] formula 2.5):

$$L_n{}^{(-a-2n+1)}(b/z) = \frac{(b/z)^n}{n!} y_n(z, a, b). \tag{1}$$

From this it follows that, on taking

$$\nu = -2n - a + 1, \qquad b = 1$$

and by the RODRIGUE's formula for y_n (z, a, b)

$$L_n{}^{\nu}(1/z) = (-1)^n e^{1/z} z^{1+n-\nu} \left(\frac{d}{dz}\right)^n (e^{-1/z} z^{-\nu-1}) \tag{2}$$

a result due to TRUESDELL [4].

N. OBRECHKOFF [3] in a very interesting preliminary note mentioned the polynomials

$$p_n(z) = e^{-1/z} \left(\frac{d}{dz}\right)^n (e^{1/z} z^{2n}). \tag{3}$$

One immediately sees that this is related to BESSEL polynomials (or LAGUERRE polynomials), with $a = b = 2$

$$y_n(2z) = (-1)^n p_n(-z). \tag{4}$$

With these introductory remarks, we pass on to the main problem. TRUESDELL's [4] list of functions satisfying his F-equation shows that

$$(5) \qquad F(z, n) = \Gamma_{(n+\nu)}(-z)^{-n-\nu-1}e^{-1/z}L_n^{\nu}(1/z).$$

(A misprint in the text is corrected here).

From (1) and (5) it follows that

$$(6) \qquad F(z, n) = z^{-2n-1-\nu}e^{-1/z}y_n(bz, 1-2n-\nu, b).$$

We shall henceforward put $a = (1 - 2n - \nu)$.

Now the characteristic of the F-equation namely

$$\frac{\partial}{\partial z}F(z, n) = F(z, n+1)$$

in the present case yields an interesting new recurrence relation

$$(7) \quad z^2y'_n(z, a, b) - (b - (a-2)z)y_n(z, a, b) = by_{n+1}(z, a-2, b).$$

(In a private communication Prof. CARLITZ informs me that this same relation is given by Dr. AL-SALAM in a paper to be published in the Duke Mathematical Journal).

The generating function for the F-equation is

$$F(z+h, \alpha) = \sum_{n=0}^{\infty} \frac{h^n}{n!} F(z, \alpha+n).$$

Using this we have

$$(8) \quad \left(\frac{z}{z+h}\right)^{2-a} e^{\frac{hb}{z(z+h)}} y_n(z+h, a, b) = \sum_{\alpha=0}^{\infty} \frac{h^\alpha}{\alpha!}\left(\frac{b}{z^2}\right)^{\alpha} y_{n+\alpha}(z, a-2\alpha, b).$$

The multiplication theorem [5] for the F-equation states that

$$F(kz, \alpha) = \sum_{n=0}^{\infty} \frac{(k-1)^n z^n}{n!} F(z, \alpha+n),$$

where k is a scalar parameter such that the right hand side of the above converges. This gives

$$(9) \quad k^{a-2}e^{\frac{(k-1)b}{kz}} y_n(kz, a, b) = \sum_{\alpha=0}^{\infty} \frac{(k-1)^\alpha}{\alpha!}\left(\frac{b}{z^2}\right)^{\alpha} y_{n+\alpha}(z, a-2\alpha, b).$$

Another generating function generated by an integral

$$\sum_{n=0}^{\infty} F(z,\ \alpha + n)\omega^n = \int_0^{\infty} e^{-t} F(z + t\omega,\ \alpha)dt$$

provided the left hand sum converges.

In the present case this gives

$$\int_0^{\infty} e^{-t+\frac{bt\omega}{z(z+t\omega)}}\left(1 + \frac{t\omega}{z}\right)^{a-2} y_n(z + \omega t,\ a,\ b)dt = \tag{10}$$

$$= \sum_{\alpha=0}^{\infty} \omega^{\alpha}\left(\frac{b}{z^2}\right)^{\alpha} y_{n+\alpha}(z,\ a - 2\alpha,\ b).$$

Taking $n = 0$, since $y_0(z,\ a,\ b) = 1$, we get

$$\left(\frac{z}{z+h}\right)^{2-a} e^{\frac{bh}{z(z+h)}} = \sum_{\alpha=0}^{\infty} \frac{h^{\alpha}}{\alpha\,!}\left(\frac{b}{z^2}\right)^{\alpha} y_{\alpha}(z,\ a - 2\alpha,\ b) \tag{11}$$

$$k^{a-2} e^{\frac{(k-1)b}{kz}} = \sum_{\alpha=0}^{\infty} \frac{(k-1)^{\alpha}}{\alpha\,!}\left(\frac{b}{z^2}\right)^{\alpha} y_{\alpha}(z,\ a - 2\alpha,\ b) \tag{12}$$

$$\int_0^{\infty} e^{-t+\frac{bt\omega}{z(z+t\omega)}}\left(1 + \frac{t\omega}{z}\right)^{a-2} dt = \sum_{\alpha=0}^{\infty} \omega^{\alpha}(b/z^2)^{\alpha} y_{\alpha}(z,\ a - 2\alpha,\ b). \tag{13}$$

In all the above $a = (1 - \nu)$.

The F-equation has the following contour integral:

$$F(z,\ n + \alpha) = \frac{\Gamma(n+1)}{2\pi i}\int_C \frac{F(y,\ \alpha)}{(y - z)^{n+1}}\, dy.$$

This gives a contour integral for $y_n(z,\ a,\ b)$

$$y_{n+\alpha}(bz,\ a - 2\alpha,\ b) = \frac{\Gamma(\alpha+1)}{2\pi i}\, e^{1/z} z^{2\alpha-a+2}\int_C \frac{e^{-1/y} y^{a-2} y_n(by,\ a,\ b)}{(y - z)^{\alpha+1}}\, dy. \tag{14}$$

Taking $n = 0$ we get $(a = 1 - \nu)$

$$y_{\alpha}(bz,\ a,\ b) = \frac{\Gamma(\alpha+1)}{2\pi i}\, e^{1/z} z^{2-a}\int_C \frac{e^{-1/y} y^{a+2\alpha-2}}{(y - z)^{\alpha+1}}\, dy, \tag{15}$$

where C is a suitable contour enclosing the point $y = z$.

The expansion (11) gives another contour integral, for by using CAUCHY's theorem we have

$$(16)\quad \begin{aligned} y_\alpha(z, a, b) &= \frac{\Gamma(\alpha+1)}{2\pi i}\left(\frac{b}{z^2}\right)^{-\alpha}\int_C \frac{e^{\frac{tb}{z(z+t)}}}{t^{\alpha+1}}\left(1+\frac{t}{z}\right)^{a+2\alpha-2} dt = \\ &= \frac{\Gamma(\alpha+1)}{2\pi i}\,\frac{z^{2-\alpha}}{b^\alpha}\int_C \frac{e^{\frac{tb}{z(z+t)}}}{t^{\alpha+1}}(z+t)^{a+2\alpha-2}dt. \end{aligned}$$

with $a = (1 - \nu)$ where C is a suitable contour enclosing the origin.

The following interesting integral is known for the solution of the F-equation:

$$\int_0^{z_0} z^{c+\alpha}F(z, \alpha)dz = e^{i\alpha\pi}\Gamma(c+\alpha+1)f(c)$$

where $Re\,(c+\alpha) > -1$, and $f(c)$ is a function of c, to be determined.

In the present case, let us take $z_0 = \infty$ so that

$$\int_0^\infty z^{c+n+a-2}e^{-1/z}y_n(bz, a, b)dz = e^{in\pi}\Gamma(n+c+1)f(c).$$

Let $n = 0$ and change z to $1/z$ so that

$$\int_0^\infty e^{-z}z^{-c-a-2n}dz = \Gamma(1-c-a-2n) = \Gamma(c+1)f(c).$$

$$(17)\quad \begin{aligned} &\int_0^\infty e^{-b/z}z^{c+n+a-2}y_n(z, a, b)dz = \\ &= \frac{\pi e^{in\pi}b^{n+c+a-1}\Gamma(n+c+1)}{\operatorname{Sin}\pi(c+a+2n)\Gamma(1+c)\Gamma(c+a+2n)}. \end{aligned}$$

This shows that this integral is valid only for $(c+a)$ not equal to zero or integer and $Re\,(c+n) > -1$.

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