Bollettino
Unione Matematica Italiana

Leonard Carlitz

On some polynomials of Tricomi.

Bollettino dell'Unione Matematica Italiana, Serie 3, Vol. **13** (1958), n.1, p. 58–64.

Zanichelli

<http://www.bdim.eu/item?id=BUMI_1958_3_13_1_58_0>

On some polynomials of Tricomi.

Nota di LEONARD CARLITZ (Durham, U. S. A.)

Sunto. - *Dalle successioni dei polinomi* $l_n(x)$, $l_n^{\alpha}(x)$ *di* TRICOMI *si deducono due polinomi ortogonali in* $(-\infty, \infty)$ *e si determinano le corrispondenti funzioni peso.*

Summary. - TRICOMI *has introduced a set of polynomials* $l_n(x)$ *and a generalized set* $l_n^{(\alpha)}(x)$ *which satisfy recurrences of the second order but are not orthogonal. In this note is shown that certain closely related sets of polynomials are orthogonal; moreover, a weight function is obtained in each case.*

1. TRICOMI [3] has introduced a set of polynomials

$$l_n(x) = \sum_{r=0}^{n} (-1)^r \binom{x}{r} \frac{x^{n-r}}{(n-r)!} \tag{1.1}$$

which satisfy the recurrence

$$(n+1)l_{n+1}(x) = nl_n(x) + xl_n(x) = 0, \quad l_0(x) = 1, \quad l_1(x) = 0. \tag{1.2}$$

It follows that $\deg l_n(x) = \left[\frac{1}{2} n\right]$, also

$$l_n(0) = 0, \quad l'_n(0) = -\frac{1}{n} \qquad (n \geq 1).$$

As TRICOMI notes. the $l_n(x)$ do not form an orthogonal set.

We should like to point out, to begin with, that a closely related set of polynomials does have the orthogonality property. Indeed if we put

$$f_n(x) = -(n+2)x^{n+2} l_{n+2}(x^{-2}), \tag{1.3}$$

then $f_n(x)$ is of degree n,

$$f_n(-x) = (-1)^n f_n(x), \tag{1.4}$$

and (1.2) becomes

$$f_{n+1}(x) - xf_n(x) + \frac{1}{n+1} f_{n-1}(x) = 0. \tag{1.5}$$

But by a theorem of FAVARD [1], (1.5) implies that the polynomials $f_n(x)$ form an orthogonal set. We shall show that they belong to the distribution $d\Psi(x)$, where $\Psi(x)$ is a step function with the jump

$$\frac{1}{2}\frac{j^{j-1}e^{-j}}{j!} \quad \text{at the point} \quad \pm j^{-\frac{1}{2}} \qquad (j = 1, 2, 3, \ldots). \tag{1.5}$$

2. We recall [2, p. 125] that

$$\sum_{n=1}^{\infty} \frac{(n+a)^n w^n}{n!} = \frac{e^{az}}{1-z} \qquad (|w| < e^{-1}), \tag{2.1}$$

where $w = ze^{-z}$. Now consider

$$n! \sum_{j=1}^{\infty} \frac{j^{j-n} w^j}{j!} l_n(j)$$

$$= n! \sum_{j=1}^{\infty} \frac{j^{j-n} w^j}{j!} \sum_{r=0}^{n} (-1)^r \binom{j}{r} \frac{j^{n-r}}{(n-r)!}$$

$$= \sum_{r=0}^{n} (-1)^r \binom{n}{r} \sum_{\substack{j=1 \\ j \geq r}}^{\infty} \frac{j^{j-r} w^j}{(j-r)!}$$

$$= \sum_{j=1}^{\infty} \frac{j^j w^j}{j!} + \sum_{r=1}^{n} (-1)^r \binom{n}{r} w^r \sum_{j=0}^{\infty} \frac{(j+r)^j w^j}{j!}$$

$$= -1 + \frac{1}{1-z} + \sum_{r=1}^{n} (-1)^r \binom{n}{r} w^r \cdot \frac{e^{rz}}{1-z}$$

$$= -1 + \frac{1}{1-z} \sum_{r=0}^{n} (-1)^r \binom{n}{r} z^r$$

$$= -1 + (1-z)^{n-1}.$$

It follows that

$$\sum_{j=1}^{\infty} \frac{j^{j-n} e^{-j}}{j!} l_n(j) = -\frac{1}{n!} \qquad (n \geq 2). \tag{2.2}$$

We observe that the series

$$\sum_{j=1}^{\infty} \frac{j^{j-n}e^{-j}}{j\,!} \qquad (n = 1, 2, 3, \ldots)$$

converges since by STIRLING's formula

$$\frac{j^{j-n}e^{-j}}{j\,!} \sim (2\pi)^{-\frac{1}{2}} j^{-n-\frac{1}{2}}.$$

Using (1.3), it is clear that (2.2) becomes

$$\sum_{j=1}^{\infty} \frac{j^{j-1}e^{-j}}{j\,!}\left(j-\tfrac{1}{2}\right)^n f_n\left(j-\tfrac{1}{2}\right) = \frac{1}{(n+1)\,!} \qquad (n \geq 0), \tag{2.3}$$

or if we prefer

$$\int_{-\infty}^{\infty} x^n f_n(x) d\Psi(x) = \frac{1}{(n+1)\,!} \qquad (n \geq 0). \tag{2.4}$$

In particular for $n = 0$, (2.4) shows that the total variation of $\Psi(x)$ is equal to 1.

It follows at once from (1.5) and (2.4) that

$$\int_{-\infty}^{\infty} x^{n-1} f_{n+1}(x) d\Psi(x) = \int_{-\infty}^{\infty} x^n f_n(x) d\Psi(x) - \frac{1}{n+1} \int_{-\infty}^{\infty} x^{n-1} f_{n-1}(x) d\Psi(x),$$

$$\int_{-\infty}^{\infty} x^{n-2} f_n(x) d\Psi(x) = 0 \qquad (n \geq 2).$$

Now assume

$$\int_{-\infty}^{\infty} x^{n-2s} f_n(x) d\Psi(x) = 0 \qquad (s = 1, \ldots, r;\ 2r < n;\ n = 3, 4, 5, \ldots);$$

then by (1.5)

$$\int_{-\infty}^{\infty} x^{n-2r-1} f_{n+1}(x) d\Psi(x) = \int_{-\infty}^{\infty} x^{n-2r} f_n(x) d\Psi(x) - \frac{1}{n+1} \int_{-\infty}^{\infty} x^{n-2r-1} f_{n-1}(x) d\Psi(x).$$

Hence we have

$$\int_{-\infty}^{\infty} x^{n-2r} f_n(x) d\Psi(x) = 0 \qquad (0 < 2r \leq n). \tag{2.5}$$

But in view of (1.4)

$$\int_{-\infty}^{\infty} x^{n-2r-1} f_n(x) d\Psi(x) = 0,$$

so that

$$\int_{-\infty}^{\infty} x^{n-r} f_n(x) d\Psi(x) = 0 \qquad (1 \leq r \leq n). \tag{2.6}$$

Since the highest coefficient in $f_n(x)$ is 1, (2.4) and (2.5) imply

$$\int_{-\infty}^{\infty} f_n^2(x) d\Psi(x) = \frac{1}{(n+1)!}. \tag{2.7}$$

Combining (2.6) and (2.7) we get

$$\int_{-\infty}^{\infty} f_n(x) f_n(x) d\Psi(x) = \frac{\delta_{mn}}{(n+1)!}, \tag{2.8}$$

so that the polynomials

$$\{(n+1)!\}^{-\frac{1}{2}} f_n(x) \qquad (n = 0, 1, 2, \ldots)$$

form an orthonormal set relative to the distribution $d\Psi(x)$.

3. In the paper referred to, TRICOMI has defined a more general polynomial

$$l_n^{(\alpha)}(x) = \sum_{r=0}^{n} (-1)^r \binom{x-\alpha}{r} \frac{x^{n-r}}{(n-r)!}, \tag{3.1}$$

so that $l_n^{(0)}(x) = l_n(x)$ and

$$(n+1) l_{n+1}^{(\alpha)} - (n+\alpha) l_n^{(\alpha)}(x) + x l_{n-1}^{(\alpha)}(x) = 0; \tag{3.2}$$

also $l_n^{(\alpha)}(x)$ is of degree $\leq \frac{1}{2} n$. If we now define

$$f_n^{(\alpha)}(x) = x^n l_n^{(\alpha)}(x^{-2}), \tag{3.3}$$

then (3.2) becomes

$$(n+1) f_{n+1}^{(\alpha)}(x) - (n+\alpha) x f_n^{(\alpha)}(x) + f_{n-1}^{(\alpha)}(x) = 0. \tag{3.4}$$

It is clear from (3.3) that

$$f_n^{(\alpha)}(-x) = (-1)^n f_n^{(\alpha)}(x). \tag{3.5}$$

Since

$$l_n^{(\alpha)}(0) = (-1)^n \binom{-\alpha}{n} = \frac{\alpha(\alpha+1)\dots(\alpha+n-1)}{n!},$$

it follows that $f_n(x)$ is of degree n provided

$$\alpha \neq 0, \ -1, \ -2, \dots . \tag{3.6}$$

Again by FAVARD's theorem, if $\alpha > 0$, il follows from (3.4) that the polynomials $f_n^{(\alpha)}(x)$ form an orthogonal set. We shall show that they belong to the distribution $d\Psi_\alpha(x)$, where $\Psi_\alpha(x)$ is a step function with the jump

$$\frac{1}{2} \alpha e^{-\alpha} \frac{(j+\alpha)^{j-1}}{j!} \tag{3.7}$$

at the points

$$\pm (j+\alpha)^{-\frac{1}{2}} \qquad (j = 0, 1, 2, \dots). \tag{3.8}$$

The condition $\alpha > 0$ ensures that the jumps (3.7) are non-negative so that the function $\Psi_\alpha(x)$ is non-decreasing.

4. In place of (2.1) we shall require

$$1 + \sum_{1}^{\infty} \frac{a(n+a)^{n-1} w^n}{n!} = e^{az}, \tag{4.1}$$

where as before $w = ze^{-z}$.

If we assume (3.6), then

$$n! \sum_{j=0}^{\infty} \frac{(j+\alpha)^{j-n-1} w^j}{j!} l_n^{(\alpha)}(j+\alpha)$$

$$= n! \sum_{j=0}^{\infty} \frac{(j+\alpha)^{j-n-1} w^j}{j!} \sum_{r=0}^{n} (-1)^r \binom{j}{r} \frac{(j+\alpha)^{n-r}}{(n-r)!}$$

$$= \sum_{r=0}^{n} (-1)^r \binom{n}{r} \sum_{j=r}^{\infty} \frac{(j+\alpha)^{j-r-1} w^j}{(j-r)!}$$

$$= \sum_{r=0}^{n} (-1)^r \binom{n}{r} w^r \sum_{j=0}^{\infty} \frac{(j+r+\alpha)^{j-1} w^j}{j!}$$

$$= \sum_{r=0}^{n} (-1)^r \binom{n}{r} w^r \cdot \frac{e^{(r+\alpha)z}}{r+\alpha}$$

$$= e^{\alpha z} \sum_{r=0}^{n} (-1)^r \binom{n}{r} \frac{z^r}{r+\alpha}.$$

Thus for $z = 1$, we get

$$n! \sum_{j=0}^{\infty} \frac{(j+\alpha)^{j-n-1} e^{-j}}{j!} l_n^{(\alpha)}(j+\alpha)$$

$$= e^{\alpha} \sum_{r=0}^{n} (-1)^r \binom{n}{r} \frac{1}{r+\alpha} = \frac{n!\, e^{\alpha}}{\alpha(\alpha+1)\dots(\alpha+n)};$$

using (3.3) this becomes

$$\text{(4.2)} \qquad \sum_{j=0}^{\infty} \frac{(j+\alpha)^{j-1} e^{-j}}{j!} (j+\alpha)^{-\frac{n}{2}} f_n^{(\alpha)}\left((j+\alpha)^{-\frac{1}{2}}\right) = \frac{e^{\alpha}}{(\alpha)_{n+1}} \; (n \geq 0),$$

where $(\alpha)_r = \alpha(\alpha+1)\dots(\alpha+r-1)$. In terms of the distribution $d\Psi_\alpha(x)$, (4.2) becomes

$$\text{(4.3)} \qquad \int_{-\infty}^{\infty} x^n f_n^{(\alpha)}(x) d\Psi_\alpha(x) = \frac{1}{(\alpha+1)_n};$$

in particular for $n = 0$, it is evident that the total variation of $\Psi(x)$ is equal to 1.

Next employing (3.4) we get

$$(n+1) \int_{-\infty}^{\infty} x^{n-1} f_{n+1}^{(\alpha)}(x) d\Psi_\alpha(x) = (n+\alpha) \int_{-\infty}^{\infty} x^n f_n^{(\alpha)}(x) d\Psi_\alpha(x)$$

$$- \int_{-\infty}^{\infty} x^{n-1} f_{n-1}^{(\alpha)}(x) d\Psi_\alpha(x),$$

so that

$$\int_{-\infty}^{\infty} x^{n-2} f_n^{(\alpha)}(x) d\Psi_\alpha(x) = 0.$$

The rest of the argument is now exactly like that of § 2; we find that

$$\int_{-\infty}^{\infty} f_m^{(\alpha)}(x) f_n^{(\alpha)}(x) d\Psi_\alpha(x) = \frac{1}{n!} \frac{\alpha}{\alpha + n} \delta_{mn}. \tag{4.4}$$

Thus the polynomials

$$\left\{ \frac{1}{n!} \frac{\alpha}{\alpha + n} \right\}^{\frac{1}{2}} f_n^{(\alpha)}(x)$$

form an orthonormal set relative to the distribution $d\Psi_\alpha(x)$.

The formula (4.4) with the usual conditions on the function $\Psi_\alpha(x)$ require that $\alpha > 0$. If however we assume only that (3.6) holds, then we can restate the result in the following form.

$$\frac{1}{2} \sum_{j=0} \left\{ f_m^{(\alpha)}\left((j+\alpha)^{-\frac{1}{2}}\right) f_n^{(\alpha)}\left((j+\alpha)^{-\frac{1}{2}}\right) + f_m^{(\alpha)}\left(-(j+\alpha)^{-\frac{1}{2}}\right) f_n^{(\alpha)}\left(-(j+\alpha)^{-\frac{1}{2}}\right) \right\} \cdot \alpha e^{-\alpha} \frac{(j+\alpha)^{j-1}}{j!} = \frac{1}{n!} \frac{\alpha}{\alpha+n} \delta_{mn}. \tag{4.5}$$

Note that when $\alpha = -k/2$, where k is an odd positive integer, then certain points (3.8) are counted twice in the sum (4.5).

REFERENCES

[1] F. Favard, *Sur les polynomes de Tchebicheff*, « Comptes rendus de l'Académie des Sciences », Paris, vol. 200 (1935), pp. 2052-2053.

[2] G. Pólya and G. Szegö, *Aufgaben und Lehrsätze aus der Analysis*, vol. 2, Berlin, 1925.

[3] F. G. Tricomi, *A class of non-orthogonal polynomials related to those of Laguerre*, « Journal d'analyse mathématique », vol. 1 (1951), pp. 209-231.