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Stable distributions and the transforms of Stieltjes and Le Roy.

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Sunto. - *Sfruttando un classico risultato di* P. LÉVY *sulla teoria dei processi stocastici, vengono studiate alcune proprietà di certe trasformazioni di* STIELTJES *e di* LE ROY *e ne vengono fatte varie applicazioni.*

Summary - *A result of* P. LÉVY, *which is now classical in the theory of stochastic processes, states that* CAUCHY'*s transcendent, defined by formula* (1) *below, does not attain negative values for real x if and only if the (positive) index* a *of* (1) *does not exceed the value* (a = 2) *belonging to a normal distribution. It is shown that appropriate transformations of this fact lead to curious results concerning the total monotony of certain* STIELTJES *and* LE ROY *transforms. There follows, among other things, a simple direct proof of a result of* W. FELLER *on* MITTAG-LEFFLER'*s* E-*functions, and an additional proof of* LÉVY'*s result is a by-product, along with a connection of all these facts with the considerations of* F. BERNSTEIN *and* G. DOETSCH *on certain* VOLTERRA *transforms.*

INTRODUCTION. - If the unit of length is suitably chosen, then, according to CAUCHY [4], pp. 99-101, any symmetric «stable» density of probability is determined by the FOURIER transform of one of the functions $\exp(-|t|^a)$, where a is a positive constant. In view of FOURIER'S inversion formula, this means that, if the corresponding density of probability is denoted by $F_a(x)/\pi$, where $-\infty < x < \infty$, then $F_a(x)$ must be of the form

$$(1) \qquad F_a(x) = \int_0^\infty \exp(-t^a) \cos xt \; dt$$

(for some $a > 0$). It was assumed for a long time that this necessary condition is sufficient as well. But (1) actually is a density of probability if and only if

$$(2) \qquad F_a(x) < 0 \quad \text{for some} \quad x = x_a$$

does *not* take place, and, while this was taken for granted by CAUCHY for every $a > 0$, F. BERNSTEIN [2] observed that (2) happens to be the case if $a = 4$ (and his method applies to every even integer

a exceeding LAPLACE'S $a = 2$). The final result is due to P. LÉVY [6], who proved that

$$(3) \qquad F_a(x) \geqq 0 \text{ for all } x \text{ if } 0 < a \leqq 2,$$

and that (2) holds for every $a > 2$ (even if $a/2$ is not an integer).

It was observed in [8] that, for every $a > 0$, CAUCHY'S transcendents $F_a(x)$ are closely connected with the transcendents $G_b(x)$ defined, for every $b > 0$, by the LAPLACE transforms

$$(4) \qquad G_b(x) = \int_0^\infty e^{-xs} \sin(s^b) ds$$

(which, of course, are convergent for

$$(5) \qquad 0 < x < \infty$$

or Re $x > 0$), and that the functions $G_b(x)$, in turn, are connected with standard entire functions occurring in the analytic continuation of power series beyond the circle of convergence ([8], p. 681). The purpose of the present note is to put (4) and its connection with (1) to explicit use in various directions, and thus to derive a few curious relations which prove to be equivalent to (3) and to the sufficiency of $a > 2$ for (2).

Actually, there will also result an additional proof of LÉVY'S fundamental result (3) itself. In this regard, the situation is as follows: If $E_\alpha(z)$ denotes MITTAG-LEFFLER'S transcendental entire function

$$(6) \qquad E_\alpha(z) = \sum_{k=0}^{\infty} z^k/\Gamma(k\alpha + 1),$$

where $\alpha > 0$, then, as observed by W. FELLER [7],

$$(7) \qquad D^n[(-1)^n E_\alpha(-x)] \geqq 0 \text{ on (5) if } 0 < \alpha \leqq 1,$$

where $D = d/dx$ and $n = 0, 1, \ldots$. According to H. POLLARD [7], FELLER'S original proof of (7) depended on (3), whereas POLLARD [7] applied contour integrations. But it turns out that if (7) is granted, then (3) follows by direct formal work. In this sense, (3) is equivalent to (7). On the other hand, there will be given for (7) a simple and elementary proof which does not involve anything like (3) but merely a trivial change of integration variables (within the *real* field).

1. When taken out of its context, the first of the facts to be proved can simply be formulated as follows: if

$$b \geqq \frac{1}{2}, \tag{8}$$

then the infinity of inequalities

$$\int^{\infty} \frac{\sin(u^b)}{(x+u)^k}\, du > 0, \quad \text{where } k = 1, 2, \dots, \tag{9}$$

holds at every point x of the half-line (5), whereas (9) is false (for some positive $x = x_b$ and an appropriate $k = k_x$) if b is any positive index violating the assumption (8).

This sharp alternative becomes more instructive if it is looked upon from the point of view of general STIELTJES transforms

$$f(x) = \int_0^{\infty} \frac{d\lambda(u)}{x+u}, \quad \text{where } d\lambda(u) \geqq 0, \tag{10}$$

on the one hand, and of general LAPLACE-STIELTJES transforms

$$f(x) = \int_0^{\infty} e^{-xt} d\mu(t), \quad \text{where } d\mu(t) \geqq 0, \tag{11}$$

on the other hand. First, it is clear that if $\lambda(u)$, where $0 \leqq u < \infty$, is any non-decreasing function corresponding to which the integral (10) is convergent for some $x > 0$, then (10) is a convergent integral for every $x > 0$, and that the function $f(x)$ which is then defined by (10) on the open half-line (5) is totally monotone, i.e.,

$$(-D)^n f(x) \geqq 0 \quad \text{on (5) for } n = 0, 1, \dots, \tag{12}$$

where $D = d/dx$. The converse is not true, the standard example being the function $f(x) = e^{-x}$; in fact, it is well-known that to this f there does not belong any λ satisfying (10), although (12) is obviously satisfied. According to the HAUSDORFF-BERNSTEIN theorem (cf. [5], p. 281), a function $f(x)$, given on (5), will possess derivatives of arbitrarily high order satisfying (12) on (5) if and only if there exists on the closed half-line $0 \leqq t < \infty$ a non-decreasing function $\mu(t)$ in terms of which it is possible to represent $f(x)$ on (5) in the form (11), the convergence of the integral (11) for every $x > 0$ being part of the statement (note that $\mu(\infty) < \infty$ is sufficient, but not necessary, to this end). In particular, the class of functions $f(x)$ on (5) which satisfy (12) or (11) is more inclusive than the class of functions (12).

All of this together contains an explanation of the situation claimed above for (9). In order to see this, notice first that if $g(t)$, where $c \leqq t < \infty$, is any positive, monotone function satisfying $g(\infty) = 0$, then the integral

$$\int_c^\infty g(t) \sin t \, dt$$

is convergent. Hence the substitution $t = u^b$ shows that the integral

$$H_b(x) = \int_0^\infty \frac{\sin(u^b)}{x+u} \, du \tag{13}$$

is convergent at every $x > 0$ whenever $b > 0$, and that the same is true of the integrals which result from (13) by successive formal differentiations. It is also seen that n formal differentiations of (13) lead to the n-th derivative of the function $H_b(x)$ (which, therefore, has derivatives of arbitrarily high order for every $x > 0$). Consequently, (9) holds on (5) for those and only those values of $b > 0$ for which (12), where $n = k - 1$, is satisfied by $f = H_b$.

Accordingly, the assertion. to be proved, is that *the function* (13) *on* (5) *is or is not totally monotone according as its* (*positive*) *index* b *does or does not satisfy the limitation* (8). In other words, $f(x) = H_b(x)$ is representable on (5) in the form (11) if and only if (8) is satisfied. In contrast, for *no* value of b is $f(x) = H_b(x)$ of the form (10). In fact, although (13) has the form of a Stieltjes transform (10), with

$$d\lambda(u) = \sin(u^b) du, \quad \text{where} \quad 0 \leqq u < \infty, \tag{14}$$

the proviso, $d\lambda(u) \geqq 0$, of (10) is violated by (14) whenever $b > 0$. Hence the assertion follows from the uniqueness theorem of the transform (10).

2. The proof of the *italicized assertion* of Section 1 will depend on the connection between (1) and (4), which can be formulated as follows (cf. [8]): If a and b is a pair of *reciprocal* positive numbers, i.e.,

$$ab = 1, \qquad (a > 0, \ b > 0), \tag{15}$$

then

$$F_a(x) = x^{-1-a} G_b(x^{-a}) \text{ on } (5). \tag{16}$$

Since the half-line (5) goes over into itself if x is replaced by x^c, where c is any non-vanishing real number, (16) is equivalent to

$$G_b(x) = x^{-1-b} F_a(x^{-b}) \text{ on } (5). \tag{17}$$

The replacement of the line $-\infty < x < \infty$ by the [open] half-line (5), which is superfluous in (1) but not in (4), involves no loss, since (1) is an even function [and since (1) remains continuous at the end-point of (5), with

$$F_a(+0) = F_a(0) = \int_0^\infty \exp(-t^a)dt = \Gamma(1 + a^{-1}) > 0 \tag{18}$$

for every $a > 0$].

In order to verify (16), apply to (1) a partial integration (at a fixed $x > 0$) so as to differentiate the factor $\exp(-t^a)$. This leads to

$$F_a(x) = a \int_0^\infty t^a \exp(-t^a) \frac{\sin xt}{xt} dt.$$

If this identity is multiplied by x and if the integration variable t is replaced by s^b/x, where $x(>0)$ is fixed and $b(>0)$ is defined by (15), it is seen that

$$xF_a(x) = x^{-a} \int_0^\infty \exp(-x^{-a}s) \sin(s^b) dt.$$

In view of the definition (4), this is equivalent to (16).

It follows that

$$G_b(x) \geqq 0 \text{ on } (5) \text{ if and only if } b \geqq \frac{1}{2}. \tag{19}$$

In fact, it is clear from (16) and (15) that (19) is equivalent to the statement that

$$F_a(x) \geqq 0 \text{ on } (5) \text{ if and only if } a \leqq 2. \tag{20}$$

But (20) is precisely LÉVY's result (3) and its converse, the existence of an $x_a > 0$ satisfying (2) if $a > 2$; cf. (18).

Incidentally, the sign of equality cannot occur for $F_a(x)$ in (20) or, equivalently, for $G_b(x)$ in (19). But this will not be needed in what follows

3. It is well-known that, subject to the proviso that the order of the two integrations involved can be interchanged, the LAPLACE transform of the LAPLACE transform of a function is the STIELTJES transform of the latter. In fact, if $x > 0$, the interior integral on the right of the formal identity

$$(21) \qquad \int_0^\infty e^{-xt}\left[\int_0^\infty e^{-ts}g(s)ds\right]dt = \int_0^\infty g(s)\left[\int_0^\infty e^{-xt}e^{-ts}dt\right]ds$$

is $\int_0^\infty e^{-yt}dt = y^{-1}$, where $y = x + s > 0$. But it is readily verified that (21) is applicable at every $x > 0$ if $g(s) = \sin(s^b)$, where $b > 0$. On the other hand, the definitions (13) and (4) show that the repeated integral on the right of (21) and the interior integral on the left of (21) then become $H_b(x)$ and $G_b(x)$ respectively. Consequently, the identity

$$(22) \qquad H_b(x) = \int_0^\infty e^{-xt}G_b(t)dt \text{ on } (5)$$

holds for every $b > 0$.

If $b > 0$ is replaced by the stricter assumption (8), then it is seen from the *first* assertion of (19) that the integral (22) is of the form (11), the proviso $d\mu(t) \geqq 0$ of (11) being equivalent to $G_b(t) \geqq 0$ (almost everywhere). Since (12) is a trivial consequence of (11) on (5), this proves that (12) is satisfied by $f(x) = H_b(x)$ whenever the value of the index b is limited by (8).

Conversely, if b has any (positive) value violating the limitation (8), then the *second* assertion of (19) shows that (22) violates condition $d\mu(t) \geqq 0$ of (11). It follows therefore from the uniqueness theorem of LAPLACE's transform on the one hand and from the existence statement of the HAUSDORFF-BERNSTEIN theorem on the other hand that (12) cannot be satisfied by $f(x) = H_b(x)$ if b is any index violating (8).

This completes the proof of the italicized statement of Section 1. i.e., both the necessity and the sufficiency of (8) for the truth of (9) on the whole of (5). [In the limiting case $k = 0$, excluded in (9), the situation is trivial but different. For if $k = 0$, then the inequality (9) reduces to

$$(23) \qquad \int_0^\infty \sin(u^b)du > 0$$

(for every x). But (23) is true only if (8) is sharpened to $b > 1$, while the integral (23) is divergent if $0 < b \leq 1$. The integral (23) is summable to 0 in the limiting case $b = 1$ and acquires a *negative* Abelian value

$$\lim_{\varepsilon \to 0} \int_0^\infty e^{-\varepsilon u} \sin(u^b) du$$

when b passes from $b > 1$ to $b < 1$].

4. In order to deal with (7), the starting point can be that integral identity for (6) which is the starting point of MITTAG-LEFFLER'S representation of the analytic continuation (if any) of power series. It is the following representation of the kernel of CAUCHY's integral formula:

$$\frac{1}{1-z} = \int_0^\infty e^{-t} E_\alpha(zt^\alpha) dt \tag{24}$$

(cf., e.g., [5], pp. 77-83 and pp. 190-191). Here t is real, α is positive, while z, instead of being arbitrary as in (6), is either within the unit circle or in the half-plane to the left of the imaginary axis.

In particular, (24) is valid on the half-line $-\infty < z < 0$. This means that

$$\int^\infty e^{-t} E_\alpha(-xt^\alpha) dt = (1+x)^{-1} \tag{25}$$

if, as always in the sequel, x varies on the open half-line (5). Actually, the verification of (25) from the definition (6) is straightforward indeed (cf. [3], p. 42), since the legitimacy of the necessary term-by-term integration follows for trivial reasons. It is also quite trivial that successive formal differentiations of (25) are legitimate (whenever $\alpha > 0$ and $x > 0$). This means that, if $n = 0, 1, \ldots,$

$$\int_0^\infty e^{-t} t^{n\alpha} D^n E_\alpha(-xt^\alpha) dt = n!/(1+x)^{n+1}, \tag{26}$$

where the D refers to the differentiation of $E_\alpha(z)$ with respect to z itself (even though $z = -xt^\alpha$).

Corresponding to (15), let

$$\alpha\beta = 1 \qquad (\alpha > 0,\ \beta > 0)$$

and replace t and x in (26) by $s = x^\beta t$ and $y = x^{-\beta}$ respectively (it being understood that $x > 0$ is fixed when t is replaced by s). Then (25) appears in the form

$$y^{\alpha n+1} \int_0^\infty e^{-ys} s^{n\alpha} D^n E_\alpha(-s^\alpha) ds = n\,! \,/\, (1 + y^{-\alpha})^{n+1},$$

where $y > 0$ is arbitrary.

5. Since

$$\frac{(y^{\alpha n+1})^{-1}}{(1+y^{-\alpha})^{n+1}} = \frac{y^{\alpha-1}}{(1+y^\alpha)^{n+1}},$$

the last relation can be written in the form

$$\int_0^\infty e^{-xt} t^{m\alpha} D^m E_\alpha(-t^\alpha) dt = \frac{m\,!\; x^{\alpha-1}}{(1+x^\alpha)^{m+1}} \tag{27}$$

if the letters y, s and n are changed to x, t and m respectively. Accordingly, (27) is an identity on (5) whenever $\alpha > 0$ and $m = 0$, $1, \ldots$.

It will now be supposed that

$$0 < \alpha < 1. \tag{28}$$

Then successive differentiations make it obvious that (12) is satisfied by $f(x) = x^{\alpha-1}$ as well as by $f(x) = (1 + x^\alpha)^{-1}$. On the other hand, the binomial rule of differentiation shows that if (12) is satisfied by $f = g$ and by $f = h$, then it is satisfied by the product $f = gh$ as well. Hence, if the latter rule is applied $m + 1$ times, then, since $m!$ is a positive constant, it follows that, for every α satisfying (28) and for every fixed $m (= 0, 1, \ldots)$, the infinity of inequalities (12) holds for the function $f(x)$ which is the quotient on the right of (27).

It now follows from the Hausdorff–Bernstein theorem that (for fixed values of α and m) the function (27) possesses on (5) a representation (11). But the uniqueness theorem of Laplace-Stieltjes transforms assures that the Hausdorff–Bernstein representation (11) of the quotient $f(x)$ on the right of (27) must be identical with the integral on the left (27). Hence, comparison of the two integral representation shows that, since $d\lambda(t) \geqq 0$ in (11),

$$D^m E_\alpha(-t^\alpha) \geqq 0, \text{ where } 0 < t < \infty. \tag{29}$$

But the D was introduced, in (26), as the symbol denoting the differentiation of $E_\alpha(z)$ with respect to z. It follows therefore from $\alpha > 0$ that (29), where (28) is assumed and $m = 0, 1, \ldots$, is equivalent to FELLER'S result (7), where $n = 0, 1, \ldots$, since the D of (7) was defined to be d/dx.

6. In view of the HAUSDORFF-BERNSTEIN theorem, (7) is equivalent to the following assertion: Corresponding to every α satisfying (28), there exists on the closed half-line $0 \leqq t < \infty$ a monotone function $\lambda_\alpha(t)$ for which

$$E_\alpha(-x) = \int_0^\infty e^{-xt} d\lambda_\alpha(t), \quad \text{where } d\lambda_\alpha(t) \geqq 0, \tag{31}$$

is an identity on (5). It is easy to conclude from the preceding deduction that a more immediate way of defining the function $\lambda_\alpha(t)$ is the following identity on (5):

$$\frac{x^{\alpha-1}}{1+x^\alpha} = \int_0^\infty e^{-xt} D_\alpha(t) dt, \tag{32}$$

where (28) *is assumed and* D_α *denotes the* LE ROY *transform of* λ_α, *i.e.*,

$$D_\alpha(x) = \int_0^\infty \exp(-x^\alpha t) d\lambda_\alpha(t) \tag{33}$$

(in this connection, cf. the comments of F. BERNSTEIN [1], pp. 50-51, on the significance of LEROY'S transformation for the problem at hand).

As seen above, (12) is satisfied by the function $f(x)$ which is the quotient on the left of (32). Hence this $f(x)$ must possess a HAUSDORFF-BERNSTEIN representation (11). What (32) accomplishes is the determination of the $d\lambda$ in (11), as follows: $d\lambda(t) = D_\alpha(t)dt$ (which, in view of the inequality in (11), implies that the function (33) is non-negative). — Note that the function on the left of (31) is a transcendental entire function, whereas the function on the left of (32) is elementary. Thus the reducibility of (31) to (32) is along the lines of the considerations of F. BERNSTEIN and DOETSCH [3], transforming transcendental into elementary relations (even though a convolution equation of VOLTERRA'S type, instead of merely a transition from LAPLACE'S to LEROY'S transform, is employed in [3], pp. 41-43).

If $m = 0$, then (27) reduces to

$$\frac{x^{\alpha-1}}{1+x^{\alpha}} = \int_0^{\infty} e^{-xt} E_{\alpha}(-t^{\alpha}) dt. \tag{34}$$

But if λ_{α} is defined by (31), then (34) shows that the assertion (32), to be verified, is equivalent to

$$E_{\alpha}(-x^{\alpha}) = D_{\alpha}(x). \tag{35}$$

Finally, the definition (33) shows that (35) is equivalent to (31).

In view of (1), it is worth mentioning that, under the assumptions (28) and (5),

$$\frac{x^{\alpha-1}}{1+x^{\alpha}} = \int_0^{\infty} J_{\alpha}(x, t) d\lambda_{\alpha}(t), \tag{36}$$

where

$$J_{\alpha}(x, t) = \int_0^{\infty} \exp(-xu - tu^{\alpha}) du. \tag{37}$$

In fact, (36) follows from (32), (33) and (37) by an application of FUBINI'S theorem.

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REFERENCES

[1] F. BERNSTEIN, *Die Übereinstimmung derjenigen beiden Summationsverfahren einer divergenten Reihe, welche von T. E. Stieltjes und E. Borel herrühren*, « Jahresbericht der Deutschen Mathematiker-Vereinigung », vol. 28 (1919), pp. 50-63.

[2] F. BERNSTEIN, *Über das Fourierintegral...*, « Mathematische Annalen », vol. 79 (1920). pp, 265-268.

[3] F. BERNSTEIN - G. DOETSCH, « *Die Integralgleichung der elliptischen Thetanullfunktion* (III) », Göttinger Nachrichten, vol. 1922, pp. 32-52.

[4] A. CAUCHY, *Oeuvres complètes*, ser. 1, vol. 12 (1900 [1853]).

[5] G. H. HARDY, *Divergent Series*, Oxford, 1949.

[6] P. LÉVY, *Théorie des erreurs, la loi de Gauss et les lois exceptionnelles*, « Bulletin de la Société Mathématique de France », vol. 52 (1924), pp. 49-85.

[7] H. POLLARD, *The completely monotonic character of the Mittag-Leffler function $E_{\alpha}(-x)$*, « Bulletin of the American Mathematical Society », vol. 54 (1948), pp. 1115-1116.

[8] A. WINTNER, *The singularities of Cauchy's distributions*, « Duke Math. Journal », vol. 8 (1941), pp. 678-681.