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On certain polynomials associated with orthogonal polynomials.

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Sunto. - *Si studia l'ortogonalità di certe classi di polinomi associati ad alcuni classici polinomi ortogonali.*

Summary. - *This is a study of the orthogonality of some polynomials associated with the classical orthogonal polynomials.*

1. Introduction. - Classical orthogonal polynomials and their corresponding functions of the second kind obey a recurrence relation of the form

$$R_n(x) - (A_n + B_n x)R_{n-1}(x) + C_n R_{n-2}(x) = 0$$

with $A_n A_{n+1} C_n > 0$. By iterating this expression, one can obtain for any three such polynomials (or functions of the second kind) of arbitrary index the relation

$$A_{r,s,t}(x)R_r(x) + B_{r,s,t}(x)R_s(x) + C_{r,s,t}(x)R_t(x) = 0$$

where the $A_{r,s,t}(x)$, $B_{r,s,t}(x)$, and $C_{r,s,t}(x)$ are termed, after PALAMÀ [9], the associated polynomials. We shall study the implications of the orthogonal type recurrence relation with the restriction $A_n A_{n+1} C_n > 0$ replaced by the weaker restriction $A_n C_n \neq 0$. We shall show that the polynomials obeying this recurrence relation are, along with their associated polynomials, orthogonal in a restricted but readily constructable sense.

In this general setting, these polynomials have been studied by NIELSEN (1918) [8], HAHN (1940) [5], and PALAMÀ (1953) [9].

2. Definition of the Associated Polynomials and a Contiguous Relation. We shall consider polynomials obeying a recurrence relation of the form

$$(1) \quad p_n(x) - (a_n + b_n x)p_{n-1}(x) + c_n p_{n-2}(x) = 0, \quad n \geq 1,$$

where $a_n c_{n+1} \neq 0$ for $n \geq 1$ and

$$p_{-1}(x) = 0,$$

$$p_0(x) = 1,$$

$$p_1(x) = a_1 x + b_1.$$

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If we add to (1) the restriction $a_n a_{n+1} c_n > 0$, then (1) becomes (FAVARD [4]) a sufficient as well as a necessary condition for the orthogonality of the $\{p_n(x)\}$. However, our results do not require that we place this restriction.

The polynomial set $\{p_n^v(x)\}$ associated with the set of polynomials $\{p_n(x)\}$ we define to be the set obeying

$$(2) \quad p_n^v(x) - (a_{n+v} + b_{n+v}x)p_{n-1}^v(x) + c_{n+v}p_{n-2}^v(x) = 0, \quad n \geq 1$$

where v is nonnegative and integral, $a_{n+v}c_{n+1+v} \neq 0$ for $n \geq 1$, and

$$(3) \quad \begin{aligned} p_{-1}^v(x) &= 0 \\ p_0^v(x) &= 1 \\ p_1^v(x) &= a_{1+v} + b_{1+v}x \\ p_2^v(x) &= b_{1+v}b_{2+v}x^2 + (a_{1+v}b_{2+v} + a_{2+v}b_{1+v})x - c_{2+v}. \end{aligned}$$

It is obvious that $p_n^0(x) = p_n(x)$.

After setting $v=0$ in (11), it will be apparent that the associated polynomials so defined are indeed the associated polynomials of the first paragraph.

We first obtain the contiguous relation (7) below.

The polynomial sets $\{p_{n-1}^{v+1}(x)\}$ and $\{p_{n-2}^{v+2}(x)\}$, except for the initial conditions, when considered as functions of n and v satisfy (2). Hence we have

$$(2) \quad p_n^v(x) - (a_{n+v} + b_{n+v}x)p_{n-1}^v(x) + c_{n+v}p_{n-2}^v(x) = 0, \quad n \geq 1,$$

$$(4) \quad p_{n-1}^{v+1}(x) - (a_{n+v} + b_{n+v}x)p_{n-2}^{v+1}(x) + c_{n+v}p_{n-3}^{v+1}(x) = 0, \quad n \geq 2,$$

$$(5) \quad p_{n-2}^{v+2}(x) - (a_{n+v} + b_{n+v}x)p_{n-3}^{v+2}(x) + c_{n+v}p_{n-4}^{v+2}(x) = 0, \quad n \geq 3.$$

Now let

$$\Phi_s^v(x) = p_s^v(x) - (a_{1+v} + b_{1+v}x)p_{s-1}^{v+1}(x) + c_{2+v}p_{s-2}^{v+2}(x), \quad s \geq 1.$$

If we multiply (4) by $-(a_{1+v} + b_{1+v}x)$ and (5) by c_{2+v} and add the resultants to (2), we see that

$$(6) \quad \Phi_n^v(x) - (a_{n+v} + b_{n+v}x)\Phi_{n-1}^v(x) + c_{n+v}\Phi_{n-2}^v(x) = 0, \quad n \geq 3.$$

But if we evaluate $\Phi_1^v(x)$ and $\Phi_2^v(x)$ by using (3) we see that they both vanish identically. Hence, from (6), $\Phi_n^v(x)$ is zero for

all positive n . That is, (PALAMÀ [9], (18)),

$$(7) \quad p_n^v(x) - (a_{1+v} + b_{1+v}x)p_{n-1}^{v+1} + c_{2+v}p_{n-2}^{v+2}(x) = 0, \quad n \geq 1.$$

3. Relations Between the Associated Polynomials and the Functions of the Second Kind. Let us first obtain a relation (11), between any three orthogonal polynomials and their associated polynomials.

From the expressions

$$(7) \quad p_n^v(x) - (a_{1+v} + b_{1+v}x)p_{n-1}^{v+1}(x) + c_{2+v}p_{n-2}^{v+2}(x) = 0, \quad n \geq 1,$$

$$(8) \quad p_m^v(x) - (a_{1+v} + b_{1+v}x)p_{m-1}^{v+1}(x) + c_{2+v}p_{m-2}^{v+2}(x) = 0, \quad m \geq 1,$$

let us eliminate $(a_{1+v} + b_{1+v}x)$. We have then

$$(9) \quad p_n^v(x)p_{m-1}^{v+1}(x) - p_{n-1}^{v+1}(x)p_m^v(x) = c_{2+v}[p_{n-1}^{v+1}(x)p_{m-2}^{v+2}(x) - p_{n-2}^{v+2}(x)p_{m-1}^{v+1}(x)].$$

The bracketed terms of this identity may be formed from the left member by shifting the indices m , n , and v to $m-1$, $n-1$, and $v+1$ respectively. Thus by iteration we may obtain

$$p_n^v(x)p_{m-1}^{v+1}(x) - p_{n-1}^{v+1}(x)p_m^v(x) = c_{2+v}c_{3+v}[p_{n-2}^{v+2}(x)p_{m-3}^{v+3}(x) - p_{n-3}^{v+3}(x)p_{m-2}^{v+2}(x)].$$

This iterative process may be continued until the polynomials (with suitable superscripts) $p_0(x) = 1$ and $p_{-1}(x) = 0$ are obtained. Let us assume that $m \geq n$. Then after iterating $(n-1)$ times, we have, (PALAMÀ [9], (16)),

$$(10) \quad p_n^v(x)p_{m-1}^{v+1}(x) - p_{n-1}^{v+1}(x)p_m^v(x) = \left[\prod_{i=1}^n c_{v+1+i} \right] p_{m-n-1}^{v+n+1}(x), \quad m \geq n \geq 1.$$

Between this expression and

$$p_n^v(x)p_{s-1}^{v+1}(x) - p_{n-1}^{v+1}(x)p_s^v(x) = \left[\prod_{i=1}^n c_{v+1+i} \right] p_{s-n-1}^{v+n+1}(x), \quad s \geq n \geq 1$$

we now eliminate $p_{n-1}^{v+1}(x)$ and thus obtain

$$p_n^v(x) \left[p_s^v(x)p_{m-1}^{v+1}(x) - p_{s-1}^{v+1}(x)p_m^v(x) \right] = \left[\prod_{i=1}^n c_{v+1+i} \right] \left[p_s^v(x)p_{m-n-1}^{v+n+1}(x) - p_m^v(x)p_{s-n-1}^{v+n+1}(x) \right], \quad m, s \geq n \geq 1.$$

The bracketed terms of the left member may be simplified if we use the identity obtained by setting $n = s$ in (10).

Thus

$$p_n^v(x) \left[\prod_{i=1}^s c_{v+1+i} \right] p_{m-s-1}^{v+s+1}(x) = \left[\prod_{i=1}^n c_{v+1+i} \right] \left[p_s^v(x) p_{m-n-1}^{v+n+1}(x) - p_m^v(x) p_{s-n-1}^{v+n+1}(x) \right]$$

or

$$(11) \quad \left[\prod_{i=n+1}^s c_{v+1+i} \right] p_n^v(x) p_{m-s-1}^{v+s+1}(x) = p_s^v(x) p_{m-n-1}^{v+n+1}(x) - p_m^v(x) p_{s-n-1}^{v+n+1}(x), \quad m \geq s \geq n \geq 1,$$

a relation between any three polynomials of an orthogonal set and their associated polynomials.

Let us next obtain a similar relation between any three functions from a set of functions of the second kind whose indices differ by integers. With each set of classical orthogonal polynomials $\{p_n(x)\}$ there is a set of functions of the second kind $\{q_n(x)\}$ than obey the same recurrence relation. Corresponding to each set of classical polynomials satisfying (1) we have thus for the associated polynomials and for the functions of the second kind where v is integral

$$(7) \quad p_n^v(x) - (a_{1+v} + b_{1+v}x) p_{n-1}^{v+1}(x) + c_{2+v} p_{n-2}^{v+2}(x) = 0, \quad n \geq 1,$$

and

$$(12) \quad q_{v+1}(x) - (a_{1+v} + b_{1+v}x) q_v(x) + c_{1+v} q_{v-1}(x) = 0, \quad v \geq 0.$$

If we eliminate $(a_{1+v} + b_{1+v}x)$ between these two identities, we obtain an expression

$$p_n^v(x) q_v(x) - c_{1+v} p_{n-1}^{v+1}(x) q_{v-1}(x) = p_{n-1}^{v+1}(x) q_{v+1}(x) - c_{2+v} p_{n-2}^{v+2}(x) q_v(x)$$

that may be iterated as was (9). Hence

$$(13) \quad p_n^v(x) q_v(x) - c_{1+v} p_{n-1}^{v+1}(x) q_{v-1}(x) = q_{v+n}(x), \quad v \geq 0, \quad n \geq 1.$$

By eliminating the terms involving $q_{v-1}(x)$ between (13) and

$$p_m^v(x) q_v(x) - c_{1+v} p_{m-1}^{v+1}(x) q_{v-1}(x) = q_{v+m}(x), \quad v \geq 0, \quad m \geq 1,$$

we arrive at

$$p_{m-1}^{v+1}(x) q_{v+n}(x) - p_{n-1}^{v+1}(x) q_{v+m}(x) = q_v(x) [p_n^v(x) p_{m-1}^{v+1}(x) - p_{n-1}^{v+1}(x) p_m^v(x)].$$

The bracketed expression may be simplified by the substitution (10). This substitution leads to the desired identity:

$$(14) \quad p_{m-1}^{\nu+1}(x)q_{\nu+n}(x) - p_{n-1}^{\nu+1}(x)q_{\nu+m}(x) = \left[\prod_{i=1}^n c_{\nu+1+i} \right] p_{m-n-1}^{\nu+n+1}(x)q_{\nu}(x),$$

$$\nu \geq 0, \quad m \geq n \geq 1.$$

4. «Finite» Orthogonality and the Lommel Polynomials. From (10) we may write

$$(15) \quad x^s p_n^{\nu}(x) \frac{p_{m-1}^{\nu+1}(x)}{p_m^{\nu}(x)} = x^s p_{n-1}^{\nu+1}(x) + \left[\prod_{i=1}^n c_{\nu+1+i} \right] \frac{x^s p_{m-n-1}^{\nu+n+1}(x)}{p_m^{\nu}(x)}, \quad m \geq n \geq 1.$$

Now the second term of this right member is a rational fraction with a zero at infinity of order $1 + n - s$ and hence its Laurent expansion about zero is a descending power series whose initial term involves x^{s-n-1} . The series converges outside any circle $\Gamma_{m,\nu}$ that contains the zeros of $p_m^{\nu}(x)$. If we integrate (15) around the contour $\Gamma_{m,\nu}$, the integral of the polynomial term of the right member of (15) vanishes. The integral of the rational fraction is zero when the residues at the origin are zero. Hence

$$(16) \quad \int_{\Gamma_{m,\nu}} x^s p_n^{\nu}(x) \frac{p_{m-1}^{\nu+1}(x)}{p_m^{\nu}(x)} dx \quad \left\{ \begin{array}{l} = 0 \text{ for } m \geq n > s \geq 0 \\ \neq 0 \text{ for } m > n = s \geq 0. \end{array} \right.$$

Notice that the weight function for this orthogonality relation can be constructed explicitly in a finite number of steps starting with nothing more than the recurrence relation of the polynomial set concerned.

POLLACZEK [10], working from the theory of continued fractions, has developed a related form of finite orthogonality.

As an instance of this «finite» orthogonality let us consider the modified Lommel polynomials

$$R_n(\nu, x) = (\nu)_n (2x)^n {}_2F_3(-n/2, (-n+1)/2; \nu, -n, 1-\nu-n; -1/x^2),$$

$$R_{-1}(\nu, x) = 0,$$

$$R_0(\nu, x) = 1,$$

$$R_1(\nu, x) = 2\nu x.$$

They satisfy the recurrence relation (see WATSON [14])

$$R_n(\nu, x) - 2x(\nu + n - 1)R_{n-1}(\nu, x) - R_{n-2}(\nu, x) = 0.$$

From the recurrence relation and from the explicit expressions for the first few polynomials it is obvious that for nonzero values of the real parameter ν , $R_n(\nu, x)$ is a polynomial in x of degree precisely n . It has been shown in [2] and [3] that they are orthogonal over a complex contour with respect to a quotient of BESSEL functions.

Again, from the recurrence relation and from the explicit expressions for the first few polynomials it is obvious that for nonzero values of the real parameter x , $R_n(\nu, x)$ is a polynomial in ν of degree precisely n . From (16) it follows that

$$\int_{\Gamma_{m,x}} \nu^s R_n(\nu, x) \frac{R_{m-1}(\nu+1, x)}{R_m(\nu, x)} d\nu \begin{cases} = 0 & \text{for } m \geq n > s \geq 0, \\ \neq 0 & \text{for } m \geq n = s \geq 0. \end{cases}$$

where $\Gamma_{m,x}$ is a contour in the ν plane that contains those values of ν for which $R_m(\nu, x)$ vanishes identically.

Now from WATSON [14], § 9.65, as m approaches infinity, $R_m(\nu, x)$ approaches the quotient of a certain gamma function and a certain BESSEL function. By invoking this limit one would like to construct a weight function in ν for the polynomials $R_n(\nu, x)$ that is independent of the size of n . This remains an interesting problem.

5. The Orthogonality of the Associated Polynomials. From (13) we may write

$$(17) \quad p_n^\nu(x) \frac{q_\nu(x)}{q_{\nu-1}(x)} = c_{1+\nu} p_{n-1}^{\nu+1}(x) + \frac{q_{\nu+n}(x)}{q_{\nu-1}(x)}, \quad \nu \geq 0, n \geq 0.$$

It is evident upon examination that (13) and (17) are true for $n=0$.

Let us suppose that $q_\nu(x)$ satisfies (12) and has, perhaps formally, an expansion in descending powers of x starting with the term in $x^{-\nu}$. Then the quotient $q_\nu(x)/q_{\nu-1}(x)$ will have, perhaps formally, the representation

$$\frac{q_\nu(x)}{q_{\nu-1}(x)} = \sum_{i=0}^{\infty} m_i^{(\nu)} x^{-i-1}.$$

The polynomials $p_n^\nu(x)$ may be written

$$p_n^\nu(x) = \sum_{k=0}^{\infty} p_{n,k}^{(\nu)} x^k$$

where $p_{n,k}^{(\nu)} = 0$ for $n < k$. The left member of (17) then appears as

$$(18) \quad \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} p_{n,k}^{(\nu)} m_i^{(\nu)} x^{-1-i-k} = \sum_{i=0}^{\infty} \sum_{k=0}^i p_{n,k}^{(\nu)} m_{i+k}^{(\nu)} x^{-1-i}.$$

Now when the right member of (17) is expanded into a series, the coefficient of x^j is not zero when $j = -n - 1$, while the coefficients will be zero for $-n < j \leq 0$. In terms of the right member of (18), we have then

$$\sum_{k=0}^i p_{n,k}^{(\nu)} m_{i+k}^{(\nu)} \begin{cases} \neq 0 & \text{for } i = n \\ = 0 & \text{for } 0 \leq i \leq n - 1. \end{cases}$$

But this is precisely the condition that the polynomials $p_n^{(\nu)}$ be orthogonal with respect to the moment sequence $\{m_k^{(\nu)}\}$.

6. The Associated LEGENDRE Polynomials. The LEGENDRE polynomials

$$P_n(x) = \frac{(2n)! x^n}{2^n (n!)^2} F\left(\frac{-n}{2}, \frac{1-n}{2}; \frac{1-2n}{2}; x^{-2}\right), \quad n \geq 0$$

and the LEGENDRE functions of the second kind

$$Q_n(x) = \frac{(n!)^2 2^n x^{-n-1}}{(2n+1)!} {}_2F_1\left(\frac{n+1}{2}, \frac{n+2}{2}; \frac{2n+3}{2}; x^{-2}\right), \quad n \geq 0$$

both satisfy the recurrence relation

$$R_n(x) - b_n x R_{n-1}(x) + c_n R_{n-2}(x) = 0, \quad n \geq 1$$

$$b_n = (2n-1)/n, \quad n \geq 1$$

$$c_n = (n-1)/n, \quad n > 1$$

$$c_1 = 1$$

provided we set $Q_{-1}(x) = 1$ and $P_{-1}(x) = 0$.

Denoting the associated LEGENDRE polynomials by $P_n(\nu, x)$, we have, from (17),

$$(19) \quad x^s P_n(\nu, x) \frac{Q_\nu(x)}{Q_{\nu-1}(x)} = c_{1+\nu} x^s P_{n-1}(\nu+1, x) + \frac{x^s Q_{\nu+n}(x)}{Q_{\nu-1}(x)}, \quad \nu \geq 0, n \geq 0.$$

The residues at the origin of the right member are zero for

$0 \leq s < n$ and the residue is not zero if $s = n$. Since the LEGENDRE functions of the second kind have neither poles nor zeros outside the unit circle, we may integrate (19) around a contour Γ that includes the unit circle and so obtain the residues at the origin. That is,

$$\int_{\Gamma} x^s P_n(\nu, x) \frac{Q_\nu(x)}{Q_{\nu-1}(x)} dx \begin{cases} = 0 & \text{for } 0 \leq s < n \\ \neq 0 & \text{for } s = n. \end{cases}$$

For the ordinary LEGENDRE polynomials we have

$$\int_{\Gamma} x^s P_n(x) Q_0(x) dx \begin{cases} = 0 & \text{for } 0 \leq s < n \\ \neq 0 & \text{for } s = n. \end{cases}$$

Let us now proceed to an explicit representation of the associated LEGENDRE polynomials.

The LEGENDRE polynomials $P_\nu(x)$ satisfy an expression of the form (12) so that we may, from (13), write

$$P_n(\nu, x) P_\nu(x) - c_{1+\nu} P_{n-1}(\nu+1, x) P_{\nu-1}(x) = P_{n+\nu}(x), \quad \nu \geq 0, n \geq 1.$$

Between this and (13) in the LEGENDRE polynomial and function form,

$$P_n(\nu, x) Q_\nu(x) - c_{1+\nu} P_{n-1}(\nu+1, x) Q_{\nu-1}(x) = Q_{n+\nu}(x), \quad \nu \geq 0, n \geq 1,$$

we may eliminate $P_{n-1}(\nu+1, x)$. We thus obtain

$$P_n(\nu, x) = \frac{P_{n+\nu}(x) Q_{\nu-1}(x) - P_{\nu-1}(x) Q_{\nu+n}(x)}{P_\nu(x) Q_{\nu-1}(x) - P_{\nu-1}(x) Q_\nu(x)}, \quad \nu \geq 0, n \geq 1.$$

But, from HOBSON [7], § 45, (76), this denominator is, for $\nu \geq 1$, merely ν^{-1} . If we consider the numerator as the difference between two LAURENT expansions, we see that only one of the LAURENT expansions has nonnegative powers of x present. That is, $p_n(\nu, x)$ must be ν times the polynomial terms of $P_{n+\nu}(x) Q_{\nu-1}(x)$. Hence we have

$$(20) \quad P_n(\nu, x) = \frac{(\nu+1/2)_n}{(\nu+1)_n} \sum_{i=0}^{\left[\frac{n}{2}\right]} \sum_{k=0}^i \frac{(-n-\nu)_{2i-2k} (\nu)_{2k} (2x)^{n-2i}}{(-n-\nu+1/2)_{i-k} (\nu+1/2)_k (i-k)! k!}, \quad \nu \geq 0.$$

The case $\nu=0$ may be verified by setting $\nu=0$ in the right member of (20) and using the fact that $(0)_{2k}$ is one if $k=0$ but is zero otherwise.

For some properties of the associated LEGENDRE polynomials, see HUMBERT [6]. Some properties of the associated HERMITE polynomials have been worked out by VARMA and MITRA in a sequence of papers that are listed in VARMA [13]. Also, see AL-SALAM [1]. TOSCANO [12] and PALAMÀ [9] have found further properties and have developed some relations for the associated LAGUERRE polynomials as well. None of these papers consider the orthogonality of the associated polynomials.

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