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Integrals involving products of E-functions, Bessel-functions and generalized hypergeometric functions.

Nota di F. M. RAGAB (al Cairo - Egitto)

Sunto. - *Gli integrali infiniti menzionati nel titolo sono calcolati per mezzo di funzioni* E *e di funzioni ipergeometriche generalizzate i cui sviluppi asintotici sono noti. Vengono anche calcolati molti integrali per mezzo di prodotti di funzioni di Bessel*

$$J_k\left[\sqrt[4]{\left\{\frac{\lambda}{z}\right\}}\right]K_k\left[\sqrt[4]{\left\{\frac{\lambda}{z}\right\}}\right].$$

Summary. - *The infinite integrals mentioned in the title are evaluated in terms of E-functions and generalized hypergeometric functions whose asymptotic expansions are known. Also many integrals are evaluated in terms of products of Bessel functions*

$$J_k\left[\sqrt[4]{\left\{\frac{\lambda}{z}\right\}}\right]K_k\left[\sqrt[4]{\left\{\frac{\lambda}{z}\right\}}\right].$$

1. - **Introductory:** In § 2 the following formula will be established.

If when $p \geq q+1$, $l \geq m+1$, $R(\alpha_r + k) > 0$, $r = 1, 2, 3, \ldots, p$, $R(2\beta_t - k) > 0$, $t = 1, 2, 3, \ldots, l$ and $|\operatorname{amp} z| < \pi$,

$$\int_0^\infty \lambda^{k-1} E(p;\ \alpha_r : q;\ \rho_s;\ \lambda) E(l;\ \beta_t : m;\ \sigma_u : z/\lambda^2) d\lambda$$

$$= 2^{\alpha_1+\alpha_2+\cdots+\alpha_p-\rho_1-\rho_2-\cdots-\rho_q+q-p-1} \cdot \pi^{\frac{1}{2}(q-p-1)}$$

$$\times \left[\pi^2 \operatorname{cosec}\left(\frac{1}{2}\pi k\right) z^{\frac{1}{2}k} E\left\{\begin{matrix}\frac{1}{2}\alpha_1, \ldots, \frac{1}{2}\alpha_p, \frac{1}{2}(\alpha_1+1), \ldots, \frac{1}{2}(\alpha_p+1), \\ \frac{1}{2}, 1-\frac{1}{2}k, \frac{1}{2}\rho_1, \ldots, \frac{1}{2}\rho_q, \frac{1}{2}(\rho_1+1), \ldots,\end{matrix}\right.\right.$$

$$\left.\begin{matrix}\beta_1 - \frac{1}{2}k, \ldots, \beta_l - \frac{1}{2}k : 4^{1+q-p}z \\ \frac{1}{2}(\rho_q+1), \sigma_1 - \frac{1}{2}k, \ldots, \sigma_m - \frac{1}{2}k\end{matrix}\right\}$$

$$+ \pi^2 \sec\left(\frac{1}{2}\pi k\right) 2^{p-q-1} z^{\frac{1}{2}k-\frac{1}{2}}$$

$$\times E\left\{\begin{matrix}\frac{1}{2}(\alpha_1+1), \ldots, \frac{1}{2}(\alpha_p+1), \frac{1}{2}(\alpha_1+2), \ldots, \frac{1}{2}(\alpha_p+2), \\ \frac{3}{2}, \frac{3}{2}-\frac{k}{2}, \frac{1}{2}(\rho_1+1), \ldots, \frac{1}{2}(\rho_q+1), \frac{1}{2}(\rho_1+2), \ldots,\end{matrix}\right.$$

$$\left.\begin{matrix}\beta_1 - \frac{1}{2}k + \frac{1}{2}, \ldots, \beta_l - \frac{1}{2}k + \frac{1}{2} : 4^{1+q-p}z \\ \frac{1}{2}(\rho_q+2), \sigma_1 - \frac{1}{2}k + \frac{1}{2}, \ldots, \sigma_m - \frac{1}{2}k + \frac{1}{2}\end{matrix}\right\}$$

$$- 2\pi^2 \operatorname{cosec}(\pi k) \cdot 2^{pk-qk-k}$$

$$\times E\left\{\begin{matrix} \frac{1}{2}(k+\alpha_1), \dots, \frac{1}{2}(k+\alpha_p), \frac{1}{2}(k+\alpha_1+1), \dots, \frac{1}{2}(k+\alpha_p+1), \\ 1+\frac{1}{2}k, \frac{1}{2}+\frac{1}{2}k, \frac{1}{2}(\rho_1+k), \frac{1}{2}(\rho_q+k), \frac{1}{2}(\rho_1+k+1), \end{matrix}\right.$$

$$\left.\begin{matrix} \beta_1, \dots, \beta_l : 4^{1+q-p}z \\ \frac{1}{2}(\rho_q+k+1), \sigma_1, \dots, \sigma_m \end{matrix}\right\}\Bigg] , \dots (1).$$

For other values of p, q, l, m the result holds if the integral is convergent.

In § 3, many particular cases of formula (1) will be considered. The following formulae will be required in the proof.

If $p \geq q+1$, $R\left(k+\frac{1}{2}\alpha_r\right) > 0$, $r = 1, 2, 3, \dots, p$, $|\operatorname{amp} z| < \pi$,

$$\int_0^\infty e^{-\lambda}\lambda^{k-1} E(p;\ \alpha_r : q;\ l_s : z\sqrt{\lambda})\, d\lambda$$

$$= 2^{\alpha_1+\alpha_2+\dots+\alpha_p-\rho_1-\rho_2-\dots-\rho_q+q-p} \cdot \pi^{\frac{1}{2}(q-p-1)}$$

$$\times \left[\pi\Gamma(k)\Gamma(1-k) E\left\{\begin{matrix} \frac{1}{2}\alpha_1, \dots, \frac{1}{2}\alpha_p, \frac{1}{2}(\alpha_1+1), \dots, \frac{1}{2}(\alpha_p+1) : 4^{1+q-p}z^2 \\ \frac{1}{2}, 1-k, \frac{1}{2}\rho_1, \dots, \frac{1}{2}\rho_q, \frac{1}{2}(\rho_1+1), \dots, \frac{1}{2}(\rho_q+1) \end{matrix}\right\}\right.$$

$$- \pi\Gamma\left(k-\frac{1}{2}\right)\Gamma\left(\frac{3}{2}-k\right)(2^{p-q-1}/z)$$

$$\times E\left\{\begin{matrix} \frac{1}{2}(\alpha_1+1), \dots, \frac{1}{2}(\alpha_p+1), \frac{1}{2}(\alpha_1+2), \dots, \frac{1}{2}(\alpha_p+2) : 4^{1+q-p}z^2 \\ \frac{3}{2}, \frac{3}{2}-k, \frac{1}{2}(\rho_1+1), \dots, \frac{1}{2}(\rho_q+1), \frac{1}{2}(\rho_1+2), \dots, \frac{1}{2}(\rho_q+2) \end{matrix}\right\}$$

$$+ \Gamma(-k)\Gamma(1+k)\Gamma\left(\frac{1}{2}-k\right)\Gamma\left(\frac{1}{2}+k\right)(4^{p-q-1}/z^2)^k$$

$$\left.\times E\left\{\begin{matrix} \frac{1}{2}\alpha_1+k, \dots, \frac{1}{2}\alpha_p+k, \frac{1}{2}(\alpha_1+1)+k, \dots, \frac{1}{2}(\alpha_p+1)+k : 4^{1+q-p}z^2 \\ 1+k, \frac{1}{2}+k, \frac{1}{2}\rho_1+k, \dots, \frac{1}{2}\rho_q+k, \frac{1}{2}(\rho_1+1)+k, \dots, \frac{1}{2}(\rho_q+1)+k \end{matrix}\right\}\right]$$

$$, \dots (2).$$

For other values of p, q, the formula holds if the integral is convergent, [I].

If $R(\alpha_{p+1}) > 0$,

$$\int_0^\infty e^{-u} u^{\alpha_{p+1}-1} E(p;\ \alpha_r : q;\ \rho_s : z/u) du = E(p+1;\ \alpha_r : q;\ \rho_s : z), \ldots \quad (3)$$

(see [2] p. 384 ex 106);

$$\frac{1}{2\pi \mathrm{i}} \int e^{\zeta} \zeta^{-\rho_{q+1}} E(p;\ \alpha_r : q;\ \rho_s : \zeta z) d\zeta = E(p;\ \alpha_r : q+1;\ \rho_s : z), \ldots \quad (4)$$

where the contour starts at $-\infty$ on the ζ — axis, passes positively round the origin and returns to $-\infty$ on the ζ — axis, the initial value of amp ζ being $-\pi$, (see [2] p. 394 ex 105).

2. - **Proof of the formula:** In formula (2), replace λ by λ^2/z^2 and then replace k by $\frac{1}{2}k$ and z by $\sqrt{(z)}$, so obtaining

$$\int_0^\infty \lambda^{k-1} E(p;\ \alpha_r : q;\ \rho_s : \lambda) E(\ ::\ z/\lambda^2) d\lambda$$

$$= 2^{\alpha_1+\alpha_2+\ldots+\alpha_p-\rho_1-\rho_2-\ldots-\rho_q+q-p-1} \cdot \pi^{\frac{1}{2}(q-p-1)}$$

$$\times \left[\pi^2 \operatorname{cosec}\left(\frac{1}{2}\pi k\right) z^{\frac{1}{2}k} E \left\{ \begin{matrix} \frac{1}{2}\alpha_1, \ldots, \frac{1}{2}\alpha_p, \frac{1}{2}(\alpha_1+1), \ldots, \frac{1}{2}(\alpha_p+1) : 4^{1+q-p} z \\ \frac{1}{2}, 1-k, \frac{1}{2}\rho_1, \ldots, \frac{1}{2}\rho_q, \frac{1}{2}(\rho_1+1), \ldots, \frac{1}{2}(\rho_q+1) \end{matrix} \right\} \right.$$

$$+ \pi^2 \sec\left(\frac{1}{2}\pi k\right) z^{\frac{1}{2}k-\frac{1}{2}}$$

$$\times E \left\{ \begin{matrix} \frac{1}{2}(\alpha_1+1), \ldots, \frac{1}{2}(\alpha_p+1), \frac{1}{2}(\alpha_1+2), \ldots, \frac{1}{2}(\alpha_p+2) : 4^{1+q-p} z \\ \frac{3}{2}, \frac{3}{2}-\frac{1}{2}k, \frac{1}{2}(\rho_1+1), \ldots, \frac{1}{2}(\rho_q+1). \frac{1}{2}(\rho_1+2), \ldots, \frac{1}{2}(\rho_q+2) \end{matrix} \right\}$$

$$- 2\pi^2 \operatorname{cosec}(\pi k) 2^{k(p-q-1)}$$

$$\left. \times E \left\{ \begin{matrix} \frac{1}{2}(k+\alpha_1), \ldots, \frac{1}{2}(k+\alpha_p), \frac{1}{2}(k+\alpha_1+1), \ldots, \frac{1}{2}(k+\alpha_p+1) : 4^{1+q-p} z \\ 1+\frac{1}{2}k, \frac{1}{2}+\frac{1}{2}k, \frac{1}{2}(\rho_1+k), \ldots, \frac{1}{2}(\rho_q+k), \frac{1}{2}(\rho_1+k+1), \ldots, \frac{1}{2}(\rho_q+k+1) \end{matrix} \right\} \right]$$

$$, \ldots \quad (4),$$

where $p \geq q+1$, $R(\alpha_r+k) > 0$, $r = 1, 2, 3, \ldots, p$ and $|\operatorname{amp} z| < \pi$.

This is a particular case of formula (1). In fact it is formula (1) with $l = m = 0$. On replacing z by z/u and applying formula (3) repeatedly; and then replacing z by ζz and applying formula (4) repeatedly; the general case can be deduced.

3. - **Applications:** We are now in a position to evaluate a large number of infinite integrals by means of (1). These integrals involves products of BESSEL functions, confluent hypergeometric functions. For the definitions and properties for the E-functions see [2] p. 352.

In (1) take $p = q = l = m = 0$, then it becomes

$$\int_0^\infty \exp\{-\lambda^2/z - 1/\lambda\}\lambda^{k-1}d\lambda =$$

$$\frac{1}{2}\Gamma\left(\frac{1}{2}k\right)z^{\frac{1}{2}k}{}_0F_2\left(;\frac{1}{2},\ 1-\frac{1}{2}k;\ -1/4z\right)$$

$$-\frac{1}{2}\Gamma\left(\frac{1}{2}k-\frac{1}{2}\right)z^{\frac{1}{2}k-\frac{1}{2}}{}_0F_2\left(;\frac{3}{2},\ \frac{3}{2}-\frac{1}{2}k;\ -1/4z\right)$$

$$+\alpha^{-k-1}\pi^{-\frac{1}{2}}\Gamma\left(-\frac{1}{2}k\right)\Gamma\left(\frac{1}{2}-\frac{1}{2}k\right){}_0F_2\left(;\ 1+\frac{1}{2}k,\ \frac{1}{2}+\frac{1}{2}k;\ -1/4z\right)$$

, ... (5),

because if $p \leq q$

$$E(p;\ \alpha_r : q;\ \rho_s : z) = \frac{\Gamma(\alpha_1)\ldots\Gamma(\alpha_p)}{\Gamma(\rho_1)\ldots\Gamma(\rho_q)}\,{}_pF_q\left(\begin{matrix}\alpha_1,\ldots,\alpha_p;\ -\frac{1}{z}\\ \rho_1,\ldots,\rho_q\end{matrix}\right) \quad , \ldots (6).$$

In (1) take $p = l = 2$, $q = m = 0$ with $\alpha_1 = \frac{1}{2} + n$, $\alpha_2 = \frac{1}{2} - n$, $\beta_1 = \frac{1}{2} + m$, $\beta_2 = \frac{1}{2} - m$; replace λ and z by 2λ and $8z$; then from the formula $\cos n\pi E\left(\frac{1}{2}+n,\ \frac{1}{2}-n :: 2z\right) = \sqrt{(2\pi z)}e^z K_n(z)$; , ... (7), (see [2] p. 351), it follows that if

$$R(1 \pm 2m - k) > 0,\quad R\left(k + \frac{1}{2} \pm n\right) > 0,\quad |\operatorname{amp} z| < \pi$$

$$\int_0^\infty \exp(\lambda + z/\lambda^2)\lambda^{k-\frac{3}{2}}K_n(\lambda)K_m(z/\lambda^2)d\lambda =$$

$$2^{\frac{1}{2}k-3}(\sqrt{\pi})^{-1}\cos n\pi\cos m\pi\cdot z^{\frac{1}{2}k-\frac{1}{2}}$$

$$\times E\left\{\begin{matrix}\frac{1}{4}+\frac{1}{2}n,\ \frac{3}{4}+\frac{1}{2}n,\ \frac{1}{4}-\frac{1}{2}n,\ \frac{3}{4}-\frac{1}{2}n,\ \frac{1}{2}+m-\frac{1}{2}k,\ \frac{1}{2}-m-\frac{1}{2}k : 2z\\ \frac{1}{4},\ 1-\frac{1}{2}k\end{matrix}\right\}$$

$$+\,2^{\frac{1}{2}k-\frac{7}{2}}\pi^{-\frac{1}{2}}\cos n\pi\cos m\pi\cdot z^{\frac{1}{2}k-1}$$

$$\times E\left\{\begin{matrix}\frac{3}{4}+\frac{1}{2}n,\ \frac{5}{4}+\frac{1}{2}n,\ \frac{3}{4}-\frac{1}{2}n,\ \frac{5}{4}-\frac{1}{2}n,\ 1+m-\frac{1}{2}k,\ 1-m-\frac{1}{2}k : 2z\\ \frac{3}{2},\ \frac{3}{2}-\frac{1}{2}k\end{matrix}\right\}$$

$$-\,2^{-2}\pi^{-\frac{1}{2}}\cos n\pi\cos m\pi\cdot z^{-\frac{1}{2}}$$

$$\times E\left\{\begin{matrix}\frac{1}{4}+\frac{1}{2}n+\frac{1}{2}k,\ \frac{3}{4}+\frac{1}{2}n+\frac{1}{2}k,\ \frac{1}{4}-\frac{1}{2}n+\frac{1}{2}k,\ \frac{3}{4}-\frac{1}{2}n+\frac{1}{2}k,\ \frac{1}{2}+m,\ \frac{1}{2}-m : 2z\\ 1+\frac{1}{2}k,\ \frac{1}{2}+\frac{1}{2}k\end{matrix}\right\}, \quad \ldots (8).$$

In (1) take $p=2$, $q=0$, $\alpha_1=\frac{1}{2}+n$, $\alpha_2=\frac{1}{2}-n$; replace λ and z by 2λ and $4z$, then from (7) and the formula

$$E\left(\frac{1}{2}-k'+m',\ \frac{1}{2}-k'-m' :: z\right)=$$

$$\Gamma\left(\frac{1}{2}-k'+m'\right)\Gamma\left(\frac{1}{2}-k'-m'\right)z^{-k'}e^{\frac{1}{2}z}\,W_{k',m'}(z) \quad , \ldots (9),$$

$$\int_0^\infty \exp\left(\lambda+\frac{z}{2\lambda^2}\right)\lambda^{k+2k'-\frac{1}{2}}K_n(\lambda)\,W_{k',m'}\left(\frac{z}{\lambda^2}\right)d\lambda=$$

$$\frac{\cos(n\pi)\operatorname{cosec}\left(\frac{1}{2}\pi k\right)}{2^{5/2}\Gamma\left(\frac{1}{2}-k'+m'\right)\Gamma\left(\frac{1}{2}-k'-m'\right)}z^{\frac{1}{2}k+k'}$$

$$\times E\left\{\begin{matrix}\frac{1+2n}{4}, \frac{3+2n}{4}, \frac{1-2n}{4}, \frac{3-2n}{4}, \frac{1}{2}-k'+m'-\frac{1}{2}k, \frac{1}{2}-k'-m'-\frac{1}{2}k : z\\ \frac{1}{2}, 1-\frac{1}{2}k\end{matrix}\right\}$$

$$+\frac{\cos(n\pi)\sec\left(\frac{1}{2}\pi k\right)}{2^{5/2}\Gamma\left(\frac{1}{2}-k'+m'\right)\Gamma\left(\frac{1}{2}-k'-m'\right)} z^{\frac{1}{2}k-\frac{1}{2}+k'}$$

$$\times E\left\{\begin{matrix}\frac{3+2n}{4}, \frac{5+2n}{4}, \frac{3-2n}{4}, \frac{5-2n}{4}, 1-k'+m'-\frac{1}{2}k, 1-k'-m'-\frac{1}{2}k : z\\ \frac{3}{2}, \frac{3}{2}-\frac{1}{2}k\end{matrix}\right\}$$

$$-\frac{\cos(n\pi)\cdot\operatorname{cosec}(\pi k)}{2^{3/2}\Gamma(1-k'+m')\Gamma\left(\frac{1}{2}-k'-m'\right)} z^{k'}$$

$$\times E\left\{\begin{matrix}\frac{1+2n+2k}{4}, \frac{3+2n+2k}{4}, \frac{1-2n+2k}{4}, \frac{3-2n+2k}{4}, \frac{1}{2}-k'+m', \frac{1}{2}-k'-m' : z\\ 1+\frac{1}{2}k, \frac{1}{2}+\frac{1}{2}k\end{matrix}\right\}, \quad \ldots (10),$$

where $R\left(k+\frac{1}{2}\pm n\right)>0$, $R(1-2k'\pm 2m'-k)>0$, $|\operatorname{amp} z|<\pi$.

Again in (1) take $l=m=0$; then from (9); if

$$R\left(k+\frac{1}{2}-k'\pm m\right)>0$$

and x is real and positive

$$\int_0^\infty \exp\left(\frac{1}{2}\lambda-\lambda^2/x\right)\lambda^{k-k'-1}W_{k',m'}(\lambda)d\lambda=$$

$$\frac{2^{-2k'-2}\pi^{\frac{1}{2}}}{\Gamma\left(\frac{1}{2}-k'+m'\right)\Gamma\left(\frac{1}{2}-k'-m'\right)\sin\frac{1}{2}\pi k} x^{\frac{1}{2}k}$$

$$\times E\left\{\begin{matrix}\frac{1}{4}(1-2k'+2m'),\ \frac{1}{4}(1-2k'-2m'),\ \frac{1}{4}(3-2k'+2m'), \\ \frac{1}{2},\ 1-\frac{1}{2}k\end{matrix}\right. \quad \left.\frac{1}{4}(3-2k'-2m') : \frac{x}{4}\right\}$$

$$\frac{2^{-2k'-2}\pi^{\frac{1}{2}}}{\Gamma\left(\frac{1}{2}-k'+m'\right)\Gamma\left(\frac{1}{2}-k'-m'\right)\sin\frac{1}{2}\pi k}\, x^{\frac{1}{2}k-\frac{1}{2}}$$

$$\times E\left\{\begin{matrix}\frac{1}{4}(3-2k'+2m'),\ \frac{1}{4}(5-2k'+2m'),\ \frac{1}{4}(3-2k'-2m'), \\ \frac{3}{2},\ \frac{3}{2}-\frac{1}{2}k\end{matrix}\right. \quad \left.\frac{1}{4}(5-2k'-2m') : \frac{x}{4}\right\}$$

$$-\frac{2^{-2k'+k-1}\pi^{\frac{1}{2}}}{\Gamma\left(\frac{1}{2}-k'+m'\right)\Gamma\left(\frac{1}{2}-k'-m'\right)\sin\pi k}$$

$$\times E\left\{\begin{matrix}\frac{1-2k'+2m'+2k}{4},\ \frac{3-2k'+2m'+2k}{4},\ \frac{1-2k'-2m'+2k}{4} \\ 1+\frac{1}{2}k,\ \frac{1}{2}+\frac{1}{2}k\end{matrix}\right. \quad \left.\frac{3-2k'-2m'+2k}{4} : \frac{x}{4}\right\}, \quad \dots (11).$$

In (1) take $p=q=l=m=1$, then from (6), it follows that if $R(k+\alpha)>0$, $R(2\beta-k)>0$ and x is real and positive

$$\int_0^\infty \lambda^{k-1} F(\alpha;\ \rho;\ -1/\lambda) F(\beta;\ \sigma;\ -\lambda^2/x)\, d\lambda = 2^{\alpha-\rho-1}\frac{\Gamma(\rho)\Gamma(\sigma)}{\sqrt{(\pi)}\Gamma(\alpha)\Gamma(\beta)}$$

$$\times\left[\frac{\pi^{\frac{1}{2}}\Gamma\left(\frac{1}{2}k\right)\Gamma\left(\frac{1}{2}\alpha\right)\Gamma\left(\frac{1}{2}\alpha+\frac{1}{2}\right)\Gamma\left(\beta-\frac{1}{2}k\right)}{\Gamma\left(\frac{1}{2}\rho\right)\Gamma\left(\frac{1}{2}\rho+\frac{1}{2}\right)\Gamma\left(\sigma-\frac{1}{2}k\right)}x^{\frac{1}{2}k}\right.$$

$$\times\, {}_3F_5\left(\begin{matrix}\frac{1}{2}\alpha,\ \frac{1}{2}\alpha+\frac{1}{2},\ \beta-\frac{1}{2}k;\ -1/4x\\ \frac{1}{2},\ 1-\frac{1}{2}k,\ \frac{1}{2}\rho,\ \frac{1}{2}\rho+\frac{1}{2},\ \sigma-\frac{1}{2}k\end{matrix}\right)$$

$$-\frac{\pi^{\frac{1}{2}}\Gamma\left(\frac{1}{2}k-\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\alpha+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\alpha+1\right)\Gamma\left(\frac{1}{2}+\beta-\frac{1}{2}k\right)}{\Gamma\left(\frac{1}{2}\rho+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\rho+1\right)\Gamma\left(\frac{1}{2}+\sigma-k\right)}x^{\frac{1}{2}k-\frac{1}{2}}$$

$$\times\, {}_3F_5\left(\begin{matrix}\frac{1}{2}\alpha+\frac{1}{2},\ \frac{1}{2}\alpha+1,\ \frac{1}{2}+\beta-\frac{1}{2}k;\ -1/4x\\ \frac{3}{2},\ \frac{3}{2}-\frac{1}{2}k,\ \frac{1}{2}\rho+\frac{1}{2},\ \frac{1}{2}\rho+1,\ \frac{1}{2}+\sigma-\frac{1}{2}k\end{matrix}\right)$$

$$+\frac{\Gamma\left(-\frac{1}{2}k\right)\Gamma\left(\frac{1}{2}-\frac{1}{2}k\right)\Gamma\left(\frac{1}{2}\alpha+\frac{1}{2}k\right)\Gamma\left(\frac{1}{2}\alpha+\frac{1}{2}k+\frac{1}{2}\right)\Gamma(\beta)}{\Gamma\left(\frac{1}{2}\rho+\frac{1}{2}k\right)\Gamma\left(\frac{1}{2}\rho+\frac{1}{2}k+\frac{1}{2}\right)\Gamma(\sigma)}$$

$$\left.\times\, {}_3F_5\left(\begin{matrix}\frac{1}{2}\alpha+\frac{1}{2}k,\ \frac{1}{2}\alpha+\frac{1}{2}k+\frac{1}{2},\ \beta;\ -1/4x\\ 1+\frac{1}{2}k,\ \frac{1}{2}+\frac{1}{2}k,\ \sigma,\ \frac{\rho+k}{2},\ \frac{\rho+k+1}{2}\end{matrix}\right)\right], \quad \ldots (12).$$

In (1) take $p=2$, $q=1$ and $l=m=1$; then if $R(k+\alpha_1)>0$, $R(k+\alpha_2)>0$, $R(2\beta-k)>0$ and $z=x$ where x is real and positive

$$\int_0^\infty \lambda^{k-1}F(\alpha_1,\ \alpha_2;\ \rho;\ -1/\lambda)F(\beta;\ \sigma;\ -\lambda^2/x)d\lambda=\frac{2^{\alpha_1+\alpha_2-\rho-2}\cdot\pi^{-1}\cdot\Gamma(\rho)\Gamma(\sigma)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\beta)}$$

$$\times\left[\frac{2^{\rho-\alpha_1-\alpha_2+1}\pi\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma\left(\beta-\frac{1}{2}k\right)\Gamma\left(\frac{1}{2}k\right)}{\Gamma(\rho)\Gamma\left(\sigma-\frac{1}{2}k\right)}x^{\frac{1}{2}k}\right.$$

$$\times {}_5F_5\left(\begin{matrix}\frac{1}{2}\alpha_1, \ \frac{1}{2}\alpha_1+\frac{1}{2}, \ \frac{1}{2}\alpha_2+\frac{1}{2}, \ \frac{1}{2}\alpha_2, \ \beta-\frac{1}{2}k; \ -\frac{1}{x}\\ \frac{1}{2}, \ 1-\frac{1}{2}k, \ \frac{1}{2}\rho, \ \frac{1}{2}\rho+\frac{1}{2}, \ \sigma\end{matrix}\right)$$

$$-\frac{2^{\rho-\alpha_1-\alpha_2+1}\pi\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)\Gamma\left(\beta+\frac{1}{2}-\frac{1}{2}k\right)\Gamma\left(\frac{1}{2}k-\frac{1}{2}\right)}{\Gamma(\rho+1)\Gamma\left(\sigma+\frac{1}{2}-\frac{1}{2}k\right)}x^{\frac{1}{2}k-\frac{1}{2}}$$

$$\times {}_5F_5\left(\begin{matrix}\frac{1}{2}\alpha_1+\frac{1}{2}, \ \frac{1}{2}\alpha_1+1, \ \frac{1}{2}\alpha_2+\frac{1}{2}, \ \frac{1}{2}\alpha_2+1, \ \frac{1}{2}+\beta-\frac{1}{2}k; \ -1/x\\ \frac{3}{2}, \ \frac{3}{2}-\frac{1}{2}k, \ \frac{1}{2}\rho+\frac{1}{2}, \ \frac{1}{2}\rho+1, \ \frac{1}{2}+\sigma-\frac{1}{2}k\end{matrix}\right)$$

$$+\frac{2^{\rho-\alpha_1-\alpha_2-k+1}\sqrt{\pi}\Gamma(k+\alpha_1)\Gamma(k+\alpha_2)\Gamma(\beta)\Gamma\left(-\frac{1}{2}k\right)\Gamma\left(\frac{1}{2}-\frac{1}{2}k\right)}{\Gamma(\rho+k)\Gamma(\sigma)}$$

$$\left.\times {}_5F_5\left(\begin{matrix}\frac{1}{2}\alpha_1+\frac{1}{2}k, \ \frac{1}{2}\alpha_1+\frac{1}{2}k+\frac{1}{2}, \ \frac{1}{2}\alpha_2+\frac{1}{2}k, \ \frac{1}{2}\alpha_2+\frac{1}{2}k+\frac{1}{2}, \ \beta; \ -1/x\\ 1+\frac{1}{2}k, \ \frac{1}{2}+\frac{1}{2}k, \ \frac{1}{2}\rho+\frac{1}{2}k, \ \frac{1}{2}\rho+\frac{1}{2}k+\frac{1}{2}, \ \sigma\end{matrix}\right)\right] \quad , \ldots (13).$$

4. - **Two formulae:** The following two formnlae will be established.

If when $p \geq q+1$, $R(\alpha_r+k)>0$, $r=1, 2, 3, \ldots, p$, $R(\rho_s)>0$, $s=1, 2, 3, \ldots, q$, $|\operatorname{amp} z|<\pi$, $R(k)>-1$.

$$\int_0^\infty \lambda^{k-1} E\left(k+1, \begin{matrix}\alpha_1, \ldots, \alpha_p : \lambda\\ \rho_1, \ldots, \rho_q\end{matrix}\right)$$

$$\times E\left\{\begin{matrix}k+\frac{1}{2}, \frac{1}{2}(\rho_1+k), \ldots, \frac{1}{2}(\rho_q+k), \frac{1}{2}(\rho_1+k+1), \ldots, \frac{1}{2}(\rho_q+k+1) : \frac{z}{\lambda^2}\\ \frac{1}{2}(\alpha_1+k), \ldots, \frac{1}{2}(\alpha_p+k), \frac{1}{2}(\alpha_1+k+1), \ldots, \frac{1}{2}(\alpha_p+k+1)\end{matrix}\right\} d\lambda$$

$$=2^{\alpha_1+\alpha_2+\cdots+\alpha_p-\rho_1-\rho_2-\cdots\rho_q+q-p-k+1}\cdot\pi^{\frac{1}{2}(q-p+1)}z^{\frac{1}{2}k}$$

$$\times J_k\left[\sqrt[4]{\left\{\frac{4^{1+p-q}}{z}\right\}}\right]K_k\left[\sqrt[4]{\left\{\frac{4^{1+p-q}}{z}\right\}}\right], \quad \dots (14).$$

For other values of p, q, the formula holds provided that the integral is convergent.

If when $p \geq q+1$, $R(k+\alpha_r-1)>0$, $r=1, 2, 3, \dots, p$, $R(\rho_t)>0$, $t=1, 2, 3, \dots, q$, $R(k)>2$, and $|\operatorname{amp} z|<\pi$;

$$\int_0^\infty \lambda^{k-2} E\left(\begin{matrix} \alpha_1, \dots, \alpha_p : \lambda \\ k+1, \rho_1, \dots, \rho_q \end{matrix}\right)$$

$$\times E\left\{\begin{matrix} k+\frac{1}{2}, \frac{1}{2}(\rho_1+k-1), \dots, \frac{1}{2}(\rho_q+k-1), \frac{1}{2}(\rho_1+k), \dots, \frac{1}{2}(\rho_q+k) : \frac{z}{\lambda^2} \\ \frac{1}{2}(\alpha_1+k-1), \dots, \frac{1}{2}(\alpha_p+k-1), \frac{1}{2}(\alpha_1+k), \dots, \frac{1}{2}(\alpha_p+k) \end{matrix}\right\} d\lambda$$

$$= 2^{\alpha_1+\alpha_2+\dots+\alpha_p-\rho_1-\rho_2-\dots-\rho_q+q-p-k+1}\,\pi^{\frac{1}{2}(q-p+1)} \cdot z^{\frac{1}{2}k-\frac{1}{2}}$$

$$\times J_{k-1}\left[\sqrt[4]{\left\{\frac{4^{1+p-q}}{z}\right\}}\right]K_{k-1}\left[\sqrt[4]{\left\{\frac{4^{1+p-q}}{z}\right\}}\right], \quad \dots (15).$$

For other values of p and q, the formula holds provided that the integral is convergent.

These two formulae will give some particular cases which will be considered in § 5.

Proof of the formulae: To prove (14); take in (1) $l=2q$, $m=2p$, with $\rho_{q+1}=k+1$ and $\beta_{p+1}=k+\frac{1}{2}$, then if $R(k)>-1$ and $R(\alpha_r+k)>0$, $r=1, 2, 3, \dots, p$, $R(\rho_t)>0$, $t=1, 2, 3, \dots, q$ and $|\operatorname{amp} z|<\pi$, $p \geq q+1$

$$\int_0^\infty \lambda^{k-1} E\left(\begin{matrix} \alpha_1, \dots, \alpha_p : \lambda \\ k+1, \rho_1, \dots, \rho_q \end{matrix}\right)$$

$$\times E\left\{\begin{matrix} k+\frac{1}{2}, \frac{1}{2}(\rho_1+k), \dots, \frac{1}{2}(\rho_q+k), \frac{1}{2}(\rho_1+k+1), \dots, \frac{1}{2}(\rho_q+k+1) : \frac{z}{\lambda^2} \\ \frac{1}{2}(\alpha_1+k), \dots, \frac{1}{2}(\alpha_p+k), \frac{1}{2}(\alpha_1+k+1), \dots, \frac{1}{2}(\alpha_p+k+1) \end{matrix}\right\} d\lambda$$

$$= 2^{\alpha_1+\alpha_2+\dots+\alpha_p-\rho_1-\rho_2-\dots-\rho_q+q-p-1} \cdot \pi^{\frac{1}{2}(q-p)}$$

$$\times\left[\frac{\pi^{\frac{1}{2}}\Gamma\left(\frac{1}{2}k\right)}{\Gamma\left(1+\frac{1}{2}k\right)} z^{\frac{1}{2}k}{}_0F_3\left(;\frac{1}{2},\ 1-\frac{1}{2}k,\ 1+\frac{1}{2}k;\ -\frac{4^{p-q-2}}{z}\right)\right.$$

$$-\frac{\pi^{\frac{1}{2}}\Gamma\left(\frac{1}{2}k-\frac{1}{2}\right)}{\Gamma\left(\frac{3}{2}+\frac{1}{2}k\right)} 2^{p-q-1}z^{\frac{1}{2}k-\frac{1}{2}}{}_0F_3\left(;\frac{3}{2},\frac{3}{2}-\frac{1}{2}k,\ \frac{3}{2}+\frac{1}{2}k;\ -\frac{4^{p-q-2}}{z}\right)$$

$$\left.+\frac{\Gamma\left(-\frac{1}{2}k\right)\Gamma\left(\frac{1}{2}-\frac{1}{2}k\right)}{\Gamma(1+k)}(2^{p-q-2})^k{}_0F_3\left(;1+\frac{1}{2}k,\frac{1}{2}+\frac{1}{2}k,\ 1+k;\ -\frac{4^{p-q-2}}{z}\right)\right]$$

by (6),

$$=2^{\alpha_1+\alpha_2+\dots+\alpha_p-\rho_1-\rho_2-\dots-\rho_q+q-p-k+1}\cdot\pi^{\frac{1}{2}(q-p+1)}\cdot z^{\frac{1}{2}}$$

$$J_k\left[\sqrt[4]{\left\{\frac{4^{1+p-q}}{z}\right\}}\right]K_k\left[\sqrt[4]{\left\{\frac{4^{1+p-q}}{z}\right\}}\right].$$

Thus (14) is proved. The proof of (15) can be deduced from (1) as (14).

5. - **Particular cases:** In (14), take $p=q=0$; then from the formula

$$E(:n+1:z)=z^{\frac{1}{2}n}J_n(2/\sqrt{z}), \qquad \dots (16),$$

if $R\left(\frac{1}{2}+k\right)>0$ and $|\operatorname{amp} z|<\pi$;

$$\int_0^\infty \lambda^{\frac{3}{2}k-1}J_k\left(\frac{2}{\sqrt{\lambda}}\right)\left(1+\frac{\lambda^2}{z}\right)^{-k-\frac{1}{2}}d\lambda=$$

$$\frac{2^{1-k}\pi^{\frac{1}{2}}z^{\frac{1}{2}k}}{\Gamma\left(k+\frac{1}{2}\right)}J_k\left[\sqrt[4]{\left(\frac{4}{z}\right)}\right]k_k\left[\sqrt[4]{\left(\frac{4}{z}\right)}\right], \qquad \dots (17),$$

which is in WATSON's book [3] p. 435 formula (5).

Also, (15) gives when $p = q = 0$, $|\operatorname{amp} z| < \pi$, $R(k) > \frac{1}{6}$,

$$\int_0^\infty \lambda^{\frac{3}{2}k-1} J_k\left(\frac{2}{\sqrt{\lambda}}\right)\left(1+\frac{\lambda^2}{z}\right)^{-k-\frac{1}{2}} d\lambda =$$

$$\frac{2^{1-k}\pi^{\frac{1}{2}} z^{\frac{1}{2}k-\frac{1}{2}}}{\Gamma\left(k+\frac{1}{2}\right)} J_{k-1}\left[\sqrt[4]{\left(\frac{4}{z}\right)}\right] K_{k-1}\left[\sqrt[4]{\left(\frac{4}{z}\right)}\right], \quad \ldots (18).$$

In (14), write x for z where x is real and positive, take $p = 3$ with $\alpha_3 = k + 1$; then if

$$R(\alpha_1 + k) > 0, \quad R(\alpha_2 + k) > 0, \quad R(k) > -\frac{1}{2}$$

$$\int_0^\infty \lambda^{k-1} E(\alpha_1, \alpha_2 :: \lambda)\, {}_0F_5\left\{; \frac{1}{2}(\alpha_1 + k), \frac{1}{2}(\alpha_2 + k), \frac{1}{2}(\alpha_1 + k + 1), \frac{1}{2}(\alpha_2 + k + 1), k + 1; -\frac{\lambda^2}{x}\right\} d\lambda$$

$$= 2^{1-2k}\Gamma(\alpha_1 + k)\Gamma(\alpha_2 + k)\Gamma(1 + k) x^{\frac{1}{2}k} J_k\left[\sqrt[4]{\left(\frac{256}{x}\right)}\right] K_k\left[\sqrt[4]{\left(\frac{256}{x}\right)}\right], \quad \ldots (19).$$

In (15), write x for z where x is real and positive, take $p = 3$ will $\alpha_3 = k + 1$; then if $R(\alpha_1 + k) > 1$, $R(\alpha_2 + k) > 1$, $R(k) > \frac{1}{6}$

$$\int_0^\infty \lambda^{k-2} E(\alpha_1, \alpha_2 :: \lambda)\, {}_0F_5\left\{\frac{1}{2}(\alpha_1 + k - 1), \frac{1}{2}(\alpha_2 + k - 1), \frac{1}{2}(\alpha_1 + k), \frac{1}{2}(\alpha_2 + k), k; -\frac{\lambda^2}{x}\right\} d\lambda$$

$$= 2^{3-2k}\Gamma(\alpha_1 + k - 1)\Gamma(\alpha_2 + k - 1)\Gamma(k) x^{\frac{1}{2}k-\frac{1}{2}} J_{k-1}\left[\sqrt[4]{\left(\frac{256}{x}\right)}\right] K_{k-1}\left[\sqrt[4]{\left(\frac{256}{x}\right)}\right], \quad \ldots (20).$$

In (19), write (2λ) for λ and take $\alpha_1 = \frac{1}{2} + n$, $\alpha_2 = \frac{1}{2} - n$; then from (7) il $R(k) > -\frac{1}{2}$, $R\left(k + \frac{1}{2} \pm n\right) > 0$,

$$\int_0^\infty \lambda^{k-\frac{1}{2}} e^{\lambda} K_n(\lambda)$$

$$\times {}_0F_5 \left\{ \frac{1}{2}\left(\frac{1}{2}+n+k\right), \frac{1}{2}\left(\frac{3}{2}+n+k\right), \frac{1}{2}\left(\frac{1}{2}-n+k\right), \frac{1}{2}\left(\frac{3}{2}-n+k\right), k+1; -\frac{\lambda^2}{x} \right\} d\lambda$$

$$= \frac{2^{\frac{1}{2}-3k}}{\sqrt{\pi}} \cos(n\pi) \Gamma\left(\frac{1}{2}+n+k\right) \Gamma\left(\frac{1}{2}-n+k\right) \Gamma(1+k) x^{\frac{1}{2}k}$$

$$J_k\left[\sqrt[4]{\left(\frac{256}{x}\right)}\right] K_k\left[\sqrt[4]{\left(\frac{256}{x}\right)}\right] \quad , \dots (21).$$

In (20), write 2λ for λ and take $\alpha_1 = \frac{1}{2}+n$, $\alpha_2 = \frac{1}{2}-n$; then from (7); if $R\left(k+\frac{1}{2} \pm n\right) > 1$, $R(k) > \frac{1}{6}$

$$\int_0^\infty \lambda^{k-\frac{5}{2}} e^{\lambda} K_n(\lambda)$$

$$\times {}_0F_5 \left\{ \frac{1}{2}\left(k+n-\frac{1}{2}\right), \frac{1}{2}\left(k+n+\frac{1}{2}\right), \frac{1}{2}\left(k-n-\frac{1}{2}\right), \frac{1}{2}\left(k-n+\frac{1}{2}\right), k; -\frac{4\lambda^2}{x} \right\} d\lambda$$

$$= 2^{\frac{7}{2}-3k} \pi^{-\frac{1}{2}} \cos(n\pi) \Gamma\left(k+n-\frac{1}{2}\right) \Gamma\left(k-n-\frac{1}{2}\right) \Gamma(k) x^{\frac{1}{2}k-\frac{1}{2}}$$

$$\times J_{k-1}\left[\sqrt[4]{\left(\frac{256}{x}\right)}\right] K_{k-1}\left[\sqrt[4]{\left(\frac{256}{x}\right)}\right], \quad \dots (22).$$

In (19), take $\alpha_1 = \frac{1}{2} - k' + m'$, $\alpha_2 = \frac{1}{2} - k' - m'$; then from (9) $\left(\frac{1}{2} - k' \pm m' + k\right) > 0$, $R(k) > -\frac{1}{2}$

$$\int_0^\infty \lambda^{k-k'-1} e^{\frac{1}{2}\lambda} W_{k', m'}(\lambda)$$

$$\times {}_0F_5\left\{;\frac{1-2k'+2m'+2k}{4}, \frac{1-2k'-2m'+2k}{4}, \frac{3-2k'+2m'+2k}{4},\right.$$

$$\left.\frac{3-2k'-2m'+2k}{4}, k+1; -\frac{\lambda^2}{x}\right\} d\lambda$$

$$= \frac{2^{1-2k}\Gamma\left(\frac{1}{2}-k'+m'+k\right)\Gamma\left(\frac{1}{2}-k'-m'+k\right)}{\Gamma\left(\frac{1}{2}-k'+m'\right)\Gamma\left(\frac{1}{2}-k'-m'\right)} x^{\frac{1}{2}k}$$

$$\times J_k\left[\sqrt[4]{\left(\frac{256}{x}\right)}\right] K_k\left[\sqrt[4]{\left(\frac{256}{x}\right)}\right], \qquad \dots (23).$$

In (20), take $\alpha_1 = \frac{1}{2} - k' + m'$, $\alpha_2 = \frac{1}{2} - k' - m'$; then if

$$R\left(k \pm m' - k' - \frac{1}{2}\right) > 0, \quad R(k) > \frac{1}{6},$$

(9) gives

$$\int_0^\infty \lambda^{k-k'-2} e^{\frac{1}{2}\lambda} W_{k', m'}(\lambda)$$

$$\times {}_0F_5\left(; \frac{2k+2m'-2k'-1}{4}, \frac{2k-2m'-2k'-1}{4}, \frac{2k+2m'-2k'+1}{4},\right.$$

$$\left.\frac{2k-2m'-2k'+1}{4}, k; -\frac{\lambda^2}{x}\right) d\lambda$$

$$= \frac{2^{3-2k}\Gamma\left(k+m'-k'-\frac{1}{2}\right)\Gamma\left(k-m'-k-\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}-k'+m'\right)\Gamma\left(\frac{1}{2}-k'-m'\right)} x^{\frac{1}{2}k-\frac{1}{2}}$$

$$\times J_{k-1}\left[\sqrt[4]{\left(\frac{256}{x}\right)}\right] K_{k-1}\left[\sqrt[4]{\left(\frac{256}{x}\right)}\right], \qquad \dots (24).$$

In (14), take $p=1$ with $\alpha_1 = \alpha$, $q=0$; then if $z=x$, where x is real and positive, $R(k) > -\frac{1}{2}$, $R(\alpha - k) > \frac{1}{2}$,

$$\int_0^\infty \lambda^{k-1} {}_1F_1\left(\alpha; k+1; \frac{-1}{\lambda}\right) {}_1F_2\left(k+\frac{1}{2}; \frac{1}{2}\alpha+\frac{1}{2}k, \frac{1}{2}\alpha+\frac{1}{2}k+\frac{1}{2}; -\frac{\lambda^2}{x}\right) d\lambda$$

$$= \frac{2^{1-2k}\pi^{\frac{1}{2}}\Gamma(1+k)\Gamma(\alpha+k)}{\Gamma(\alpha)\Gamma\left(k+\frac{1}{2}\right)} x^{\frac{1}{2}k} J_k\left[\sqrt[4]{\left(\frac{16}{x}\right)}\right] K_k\left[\sqrt[4]{\left(\frac{16}{x}\right)}\right], \qquad \dots (25).$$

In (15), write x for z where x is real and positive, take $p=1$ with $\alpha_1=\alpha$ and $p=0$; then if $R(k)>\frac{1}{6}$, $R(\alpha-k)>\frac{1}{2}$, $R(\alpha+k)>1$,

$$\int_0^\infty \lambda^{k-2}\,{}_1F_1\begin{pmatrix}\alpha;\ -\frac{1}{\lambda}\\ k+1\end{pmatrix}\,{}_1F_2\begin{pmatrix}k+\frac{1}{2};\ -\lambda^2/x\\ \frac{1}{2}\alpha+\frac{1}{2}k-\frac{1}{2},\ \frac{1}{2}\alpha+\frac{1}{2}k\end{pmatrix}d\lambda=$$

$$\frac{2^{2-2k}\pi^{\frac{1}{2}}\Gamma(1+k)\Gamma(\alpha+k-1)\cdot x^{\frac{1}{2}k-\frac{1}{2}}}{\Gamma(\alpha)\Gamma\left(k+\frac{1}{2}\right)}J_{k-1}\left[\sqrt[4]{\left(\frac{16}{x}\right)}\right]K_{k-1}\left[\sqrt[4]{\left(\frac{16}{x}\right)}\right], \quad \dots (26).$$

In (14), take $p=q=1$; then if $R(k)>-\frac{1}{2}$, $R(\rho)>0$, $R(\alpha+k)>0$, $R(3k+\rho-\alpha)>-\frac{1}{2}$ and x is real and positive,

$$\int_0^\infty \lambda^{k-1}\,{}_1F_2\begin{pmatrix}\alpha,\ -1/\lambda\\ 1+k,\ \rho\end{pmatrix}{}_3F_2\begin{pmatrix}k+\frac{1}{2},\ \frac{1}{2}\rho+\frac{1}{2}k,\ \frac{1}{2}\rho+\frac{1}{2}k+\frac{1}{2};\ -\lambda^2/x\\ \frac{1}{2}\alpha+\frac{1}{2}k,\ \frac{1}{2}\alpha+\frac{1}{2}k+\frac{1}{2}\end{pmatrix}d\lambda$$

$$=\frac{2^{1-k}\pi^{\frac{1}{2}}\Gamma(k+1)\Gamma(\alpha+k)\Gamma(\rho)}{\Gamma(\alpha)\Gamma\left(k+\frac{1}{2}\right)\Gamma(\rho+k)}x^{\frac{1}{2}k}J_k\left[\sqrt[4]{\left(\frac{4}{x}\right)}\right]K_k\left[\sqrt[4]{\left(\frac{4}{x}\right)}\right], \quad \dots (27),$$

In (15), take $p=q=1$; then if $R(k)>\frac{1}{6}$, $R(\rho)>0$, $R(k+\alpha)>1$, $R(3k+\rho-\alpha)>\frac{3}{2}$ and x is real and posititve

$$\int_0^\infty \lambda^{k-2}\,{}_1F_2\begin{pmatrix}\alpha;\ -1/\lambda\\ k+1,\ \rho\end{pmatrix}{}_3F_2\begin{pmatrix}k+\frac{1}{2},\ \frac{1}{2}\rho+\frac{1}{2}k-\frac{1}{2},\ \frac{1}{2}\rho+\frac{1}{2}k;\ -\lambda^2/x\\ \frac{1}{2}\alpha+\frac{1}{2}k-\frac{1}{2},\ \frac{1}{2}\alpha+\frac{1}{2}k\end{pmatrix}d\lambda$$

$$=\frac{2^{1-k}\pi^{\frac{1}{2}}\Gamma(k+1)\Gamma(\alpha+k-1)\Gamma(\rho)}{\Gamma(\alpha)\Gamma\left(k+\frac{1}{2}\right)\Gamma(\alpha+k-1)}x^{\frac{1}{2}k-\frac{1}{2}}J_{k-1}\left[\sqrt[4]{\left(\frac{4}{x}\right)}\right]K_{k-1}\left[\sqrt[4]{\left(\frac{4}{x}\right)}\right], \quad \dots (28).$$

In (14), take $p=2$, $\rho=0$; then if x is real and positive $R(\alpha_1+k)>0$, $R(\alpha_2+k)>0$, $R(k)>-\frac{1}{2}$

$$\int_0^\infty \lambda^{k-1}{}_2F_1\left(\begin{matrix}\alpha_1, \alpha_2; & -1/\lambda\\ k+1\end{matrix}\right)$$

$${}_1F_4\left(\begin{matrix}k+\frac{1}{2}; \quad -\lambda^2/x\\ \frac{1}{2}\alpha_1+\frac{1}{2}k, \frac{1}{2}\alpha_1+\frac{1}{2}k+\frac{1}{2}, \frac{1}{2}\alpha_2+\frac{k}{2}, \frac{\alpha_2}{2}+\frac{k}{2}+\frac{1}{2}\end{matrix}\right)d\lambda$$

$$=\frac{2^{1-3k}\pi^{\frac{1}{2}}\Gamma(k+1)\Gamma(\alpha_1+k)\Gamma(\alpha_2+k)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma\left(k+\frac{1}{2}\right)}x^{\frac{1}{2}k}J_k\left[\sqrt[4]{\left(\frac{64}{x}\right)}\right]K_k\left[\sqrt[4]{\left(\frac{64}{x}\right)}\right], \dots (29).$$

Similary (15) with $p=2$, $q=0$, gives if x is real and positive, $R(\alpha_1+k)>1$, $R(\alpha_2+k)>1$, $R(\lambda)>\frac{1}{6}$,

$$\int_0^\infty \lambda^{k-2}{}_2F_1\left(\begin{matrix}\alpha_1, \alpha_2; & -1/\lambda\\ k+1\end{matrix}\right)$$

$${}_1F_4\left(\begin{matrix}k+\frac{1}{2}; \quad -\lambda^2/x\\ \frac{1}{2}\alpha_1+\frac{k}{2}-\frac{1}{2}, \frac{1}{2}\alpha_1+\frac{1}{2}k, \frac{\alpha_2}{2}+\frac{k}{2}-\frac{1}{2}, \frac{\alpha_2}{2}+\frac{k}{2}\end{matrix}\right)d\lambda$$

$$=\frac{2^{3-2k}\pi^{\frac{1}{2}}\Gamma(k+1)\Gamma(\alpha_1+k-1)\Gamma(\alpha_2+k-1)}{\Gamma\left(k+\frac{1}{2}\right)\Gamma(\alpha_1)\Gamma(\alpha_2)}x^{\frac{1}{2}k-\frac{1}{2}}$$

$$\times J_{k-1}\left[\sqrt[4]{\left(\frac{64}{x}\right)}\right]K_{k-1}\left[\sqrt[4]{\left(\frac{64}{x}\right)}\right], \dots (30).$$

In (14), take $p=2$, $q=1$, and get

$$\int_0^\infty \lambda^{k-1}{}_2F_2\left(\begin{matrix}\alpha_1, \alpha_2, & -1/\lambda\\ k+1, \rho\end{matrix}\right)$$

$$\times {}_3F_4\left(\begin{matrix} k+\frac{1}{2},\ \frac{1}{2}\rho+\frac{1}{2}k,\ \frac{1}{2}\rho+\frac{1}{2}k+\frac{1}{2};\ -\lambda^2/x \\ \frac{1}{2}\alpha_1+\frac{1}{2}k,\ \frac{1}{2}\alpha_1+\frac{1}{2}k+\frac{1}{2},\ \frac{1}{2}\alpha_2+\frac{1}{2}k,\ \frac{1}{2}\alpha_2+\frac{1}{2}k+\frac{1}{2} \end{matrix}\right)d\lambda$$

$$=\frac{2^{1-2k}\pi^{\frac{1}{2}}\Gamma(\alpha_1+k)\Gamma(\alpha_2+k)\Gamma(k+1)\Gamma(\rho)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\rho+k)\Gamma\left(k+\frac{1}{2}\right)}x^{\frac{1}{2}k}J_k\left[\sqrt[4]{\left(\frac{16}{x}\right)}\right]K_k\left[\sqrt[4]{\left(\frac{16}{x}\right)}\right] \quad ,\ \dots\ (31),$$

where $R(k)>-\frac{1}{2}$, $R(\rho)>0$, $R(k+\alpha_1)>0$, $R(k+\alpha_2)>0$, and x is real and positive.

In (15), take $p=2$, $q=1$, and get

$$\int_0^\infty \lambda^{k-2}\,{}_2F_2\left(\begin{matrix}\alpha_1,\ \alpha_2;\ -\frac{1}{\lambda} \\ k+1,\ \rho\end{matrix}\right)$$

$$\times {}_3F_4\left\{\begin{matrix} k+\frac{1}{2},\ \frac{1}{2}(k+\rho-1),\ \frac{1}{2}(k+\rho);\ -\lambda^2/x \\ \frac{1}{2}(\alpha_1+k-1),\ \frac{1}{2}(\alpha_1+k),\ \frac{1}{2}(\alpha_2+k-1),\ \frac{1}{2}(\alpha_2+k) \end{matrix}\right\}d\lambda$$

$$=\frac{2^{2-2k}\pi^{\frac{1}{2}}\Gamma(\alpha_1+k-1)\Gamma(\alpha_2+k-1)\Gamma(k+1)\Gamma(\rho)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma\left(k+\frac{1}{2}\right)\Gamma(\rho+k-1)}x^{\frac{1}{2}k-\frac{1}{2}}$$

$$\times J_{k-1}\left[\sqrt[4]{\left(\frac{16}{x}\right)}\right]K_{k-1}\left[\sqrt[4]{\left(\frac{16}{x}\right)}\right] \quad ,\ \dots\ (32),$$

where $R(\lambda)>\frac{1}{2}$, $R(\rho)>0$, $R(\alpha_1+k)>1$, $R(\alpha_2+k)>1$ and x is real and positive.

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