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On the non-negativity of solutions of the heat equation.

Nota di Richard Bellman (a Santa Monica - California)

Sunto. - *Viene mostrato come una combinazione di classici teoremi di esistenza e unicità e di metodi relativi a differenze finite fornisce una semplicissima dimostrazione di non negatività delle soluzioni dell'equazione del calore.*

Summary. - *It is shown that a combination of classical existence and uniqueness theorems and finite difference techniques yields a very simple proof of non-negativity.*

1. - Introduction.

In treating a functional equation, once the question of existence and uniqueness has been disposed of, we turn to a more precise study of the analytic character of the solution. It frequently happens that a method which works very efficiently to establish existence and uniqueness does not yield other properties of the solution in any ready fashion. Conversely, methods which yield non-negativity, convexity, and so forth, may not be ideally suited for the establishment of the basic properties. However, a combination of several techniques may yield the results we desire quite easily.

illustrate these remarks, let us consider the heat equation

$$
\begin{aligned}
&u_t = u_{xx} + q(x, t)u, \\
(1) \qquad &u(x, 0) = v(x), \ 0 \leq x \leq 1, \\
&u(0, t) = u(1, t) = 0, \ t > 0.
\end{aligned}
$$

We shall assume that we have demonstrated, by some means or other, the existence of a solution which depends continuously upon $v(x)$ in the L^2-norm for $t \geq 0$, and that this solution is unique. As we shall see, a method based upon finite differences will enable us to demonstrate the fact that this solution is non-negative for $t \geq 0$, provided that it is non-negative at $t = 0$, i.e. provided that $v(x) \geq 0$, for $0 \leq x \leq 1$. On the other hand, the particular proof used to establish existence and uniqueness may not have yielded non-negativity in a simple fashion, and, as is known, an existence and uniqueness proof based upon finite differences is not a completely simple matter.

2. · $u_t = u_{xx}$ ·

To illustrate our ideas, begin with the simpler equation

$$u_t = u_{xx} \tag{1}$$

and consider the difference scheme

$$(2) \quad \begin{aligned} &(a) \quad w(x,\ t + \delta^2/2) = [w(x + \delta,\ t) + w(x - \delta,\ t)]/2, \\ &(b) \quad w(x,\ 0) = v(x), \end{aligned}$$

where x takes the values δ, $2\delta, \ldots,$ 1, and t assumes the values 0, $\delta^2/2$, $\delta^2, \ldots$. The function $w(x,\ t)$ is defined by linearity at non-lattice points.

It is easy to see that, formally, the recurrence relation approaches the partial differential equation as $\delta \to 0$.

As mentioned above, a rigorous proof that the solution of (2) converges to the solution of (1) as $\delta \to 0$, starting from first principles, is non-trivial. However, as we shall see below, a proof of this fact is quite simple, once we have established the existence and uniqueness of a solution. The fact that $w(x,\ t)$ is non-negative for any $\delta > 0$ is immediate, and this yields the conclusion that $u(x,\ t) \geq 0$.

Since we have assumed the existence of a solution of (1) which is a continuous function of $v(x)$, there is no loss of generality in assuming, for our current purposes, that $v(x)$ possesses appropriate continuity properties, sufficient to ensure that

$$\underset{R}{\mathrm{Max}}\ [\,|\,u_{tt}(x,\ t)\,|,\ \ |\,u_{xxxx}(x,\ t)\,|\,] \leq m < \infty, \tag{3}$$

where R is the bounded region $0 \leq x \leq 1$, $0 \leq t \leq T < \infty$. We may for example take $v(x)$ to be a trigonometric polynomial. Under the assumption of (3), it is easy to show that in R

$$\lim_{\delta \to 0} w(x,\ t) = u(x,\ t). \tag{4}$$

We have, by virtue of (2)

$$u(x,\ t + \delta^2/2) = [u(x + \delta,\ t) + u(x - \delta,\ t)]/2 + \delta^4 r(x,\ t), \tag{5}$$

where $|\,r(x,\ t)\,| \leq 2m$ in R. Consequently, the function $z(x,\ t) = w(x,\ t) - u(x,\ t)$ satisfies the recurrence relation of (1a) with the initial condition $z(x,\ t) = 0$. Let

$$d(t) = \underset{0 \leq x \leq 1}{\mathrm{Max}}\ |\,w(x,\ t) - u(x,\ t)\,|. \tag{6}$$

Using the recurrence relation, we see that

(7) $$d(t+\delta^2/2) \leq d(t) + 2\delta^4 m$$

for $t = 0,\ \delta^2/2,\ \delta^2, \dots,$ whence

(8) $$d(t) \leq 2\delta^4 m N$$

for $0 \leq t \leq \delta^2 N$. Let $\delta^2 N = T$. Then

(9) $$d(t) \leq 2\delta^2 m T$$

for x and t in R.

From this, we obtain the desired result as $\delta \to 0$.

Note that we can also conclude from the foregoing result that $u(x, t)$ is concave in x for any value of t if $v(x)$ is concave in x. This, in turn, implies that $u(x, t)$ is decreasing in t for each fixed value of x.

3. $u_t = u_{xx} + q(x, t)u$.

To extend the same argument to the general equation of (1.1), we employ the recurrence relation

(1) $$w(x, t+\delta^2/2) = \frac{w(x+\delta, t) + w(x-\delta, t)}{2} + \int_{x-q(x,t)\delta^2/2}^{x+q(x,t)\delta^2/2} w(y, t)\,dy.$$

If we assume that $q(x, t) \geq 0$ for $0 \leq x \leq 1$, $t \geq 0$, the recurrence relation above shows that $w(x, t) \geq 0$ for all x and t. The proof that $w(x, t)$ converges to $u(x, t)$ as $\delta \to 0$ follows the same lines as before.

To see that it is sufficient to assume that

(2) $$q(x, t) \geq -\lambda > -\infty, \quad 0 \leq x \leq 1, \quad 0 \leq t \leq T,$$

for any $T > 0$, where $\lambda = \lambda(T)$, we proceed as follows. Write

(3) $$u = e^{-\lambda t} v.$$

Then the equation of (1.1) becomes

(4) $$v_t = v_{xx} + (q(x, t) + \lambda)v,$$

with the same boundary conditions. The new function

$$q_1(x,\ t) = q(x,\ t) + \lambda \tag{5}$$

is non-negative.

4. - Generalizations.

It is clear that the same method may be employed to obtain corresponding non-negativity results for the solution of the heat equation for higer dimensions and arbitrary regions. The essential part of the proof is the à priori demonstration of the existence and uniqueness of a solution depending continuously upon the initial values, in an appropriate metric.

Similary, a number of corresponding results can be established for various classes of non linear equations.