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A General Definition of Convergence, Continuity, Differentiability and Integrability.

Nota di SMBAT ABIAN (a Knoxville, Tennessee, U. S. A.)

Sunto. - *In topologia, la nozione di convergenza e i concetti relativi, sono studiati per mezzo di filtri e reti. Qui si è tentato di introdurre queste idee negli spazi astratti nei quali è definita la struttura più semplice possibile.*

Summary. - *In topology the notion of convergence and related concepts are studied by means of filters and nets. Here, an attempt is made to introduce these ideas in abstract spaces on which the simplest possible structure is defined.*

1. **Structure on a set.** - A collection

$$\mathfrak{X} = (X_i)_{i \in I} \tag{1}$$

of subsets of a set X is defined to be a structure on X if

$$\bigcup_{i \in I} X_i = X. \tag{2}$$

2. **Structural element.** - Every X_i in (2) is defined to be a structural element of the structure $\mathfrak{X}$ in (1).

In what follows the set X will have the structure $\mathfrak{X}$ defined on it as indicated in (1). An arbitrary letter P will indicate a set with a structure $\mathfrak{P}$ whose structural elements are P_i, i belonging to an appropriate index set.

3. **Convergent family.** - A family $\mathfrak{F} = (X_i)_{i \in J \subset I}$ of structural elements of the structure $\mathfrak{X}$ is defined to be a convergent family, converging to $\bigcap_{i \in J} X_i = L \neq \emptyset$, if $\mathfrak{F}$ consists of all of those structural elements which intersect L, i. e.,

$$\mathfrak{F} = (X_i)_{i \in J}, \quad \bigcap_{i \in J} X_i = L \neq \emptyset, \quad \text{and} \quad J = [i \in I \mid X_i \cap L \neq \emptyset].$$

We indicate this by $\mathfrak{F} \to L$, and we call L the limit of the convergent family $\mathfrak{F}$. Also, if x represents any arbitrary element of X, we may write $x \to L$ instead of $\mathfrak{F} \to L$.

4. **Limit of a subset.** – A subset $L \subset X$ is defined to be a limit of a subset $A \subset X$, if there exists a convergent family $\mathfrak{F} = (X_i)_{i \in J} \to L$ such that

$$(X_i - L) \cap A \neq \emptyset, \qquad i \in J.$$

5. **Convergent subset.** – A subset $A \subset X$ is defined to be a convergent subset converging to L, if L is the unique limit of A. We indicate this by $A \to L$, or $x \to L$, $x \in A$.

6. **Limit of a function.** – Let f be a function (single or multivalued) mapping X into Y. A subset $M \subset Y$ is defined to be a limit of f when $x \to L$, if there exists a convergent family $\mathfrak{D} = (Y_i)_{i \in K} \to M$ such that

$$L \cup f^{-1}(Y_i) \in \mathfrak{F} \to L, \qquad i \in K,$$

and we indicate this by $M = \text{a} \lim\limits_{x \to L} f$. If M is the unique limit of f when $x \to L$, then we write $M = \lim\limits_{x \to L} f$.

7. **Continuous function.** – Let f be a function mapping X into Y. The function f is defined to be continuous on a subset $L \subset X$, if $\lim\limits_{x \to L} f = f(L)$. If f is continuous on every subset $A \subset X$ for which we can write $x \to A$, then f is defined to be continuous on X.

8. **Differentiable function.** – Let the three sets X, Y, Q and a fixed function g mapping $X \times Y$ into Q and a function f mapping X into Y be given. Let h represent the restriction of g to $[x, f(x)]$, $x \in X$. If

$$\lim_{x \to L} pr_X^{-1} h = M,$$

then M is defined to be the derivative of f on the subset $L \subset X$. We indicate this by $f'(L) = M$. If $f'(A)$ exists on every subset $A \subset X$ for which we can write $x \to A$, then f is defined to be differentiable on X.

9. Integrable function. – Let the three sets X, Y, S and a fixed function g mapping $X \times S$ into Y and a function f mapping X into Y be given. If there exists a function t mapping X into S such that

$$t'(L) = \lim_{x \to L} f,$$

then $t(L)$ is defined to be the integral of f on the subset $L \subset X$. We indicate this by $\int f(L) = t(L)$. If $\int f(A)$ exists on every subset $A \subset X$ for which we can write $x \to A$ then f is defined to be integrable on X.

REMARK. – The concepts of limit, convergence, continuity, differentiability and integrability in classical analysis and topological spaces are special cases of the same concepts as introduced above.